On operator representations of locally definitizable functions

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Abstract. Let Ω be some domain in $\overline{\mathbf{C}}$ symmetric with respect to the real axis and such that $\Omega \cap \overline{\mathbf{R}} \neq \emptyset$ and the intersections of Ω with the upper and lower open half-planes are simply connected. We study the class of piecewise meromorphic $\mathbf{R}$-symmetric operator functions G in $\Omega \setminus \overline{\mathbf{R}}$ such that for any subdomain Ω' of Ω with $\overline{\Omega'} \subset \Omega$, G restricted to Ω' can be written as a sum of a definitizable and a (in Ω') holomorphic operator function. As in the case of a definitizable operator function, for such a function G we define intervals $\Delta \subset \mathbf{R} \cap \Omega$ of positive and negative type as well as some "local" inner products associated with intervals $\Delta \subset \mathbf{R} \cap \Omega$.

Representations of G with the help of linear operators and relations are studied, and it is proved that there is a representing locally definitizable selfadjoint relation A in a Krein space which locally exactly reflects the sign properties of G: The ranks of positivity and negativity of the spectral subspaces of A coincide with the numbers of positive and negative squares of the "local" inner products corresponding to G.

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1. Introduction

Let $(\mathcal{H}, [\cdot,\cdot])$ be a separable Krein space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators in $\mathcal{H}$. Recall that a piecewise meromorphic $\mathcal{L}(\mathcal{H})$-valued function G in $\mathbf{C} \setminus \mathbf{R}$ symmetric with respect to $\mathbf{R}$ (that is, $G(\overline{z}) = G(z)^+$ for all points z of holomorphy of G; " $^+$ " denotes the Krein space adjoint) is called *definitizable* if there exists an $\mathbf{R}$-symmetric scalar rational function r such that the

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product rG is the sum of a Nevanlinna function N and an $\mathcal{L}(\mathcal{H})$-valued rational function P with the poles of P being points of holomorphy of G:

$$r(z)G(z) = N(z) + P(z)$$

for all points $z \in \mathbf{C} \setminus \mathbf{R}$ of holomorphy of rG. A rational operator function is by definition a meromorphic operator function in $\overline{\mathbf{C}}$ ([5]). The classes $N_k(\mathcal{L}(\mathcal{H}))$, $k = 0, 1, \ldots$, of generalized Nevanlinna operator functions, introduced and first studied by M. G. Krein and H. Langer, are contained in the set of the definitizable operator functions ([4], [5]).

By [5, Proposition 3.3] an $\mathbf{R}$-symmetric piecewise meromorphic $\mathcal{L}(\mathcal{H})$-valued function G in $\mathbf{C} \setminus \mathbf{R}$ is definitizable if and only if it has no more than a finite number of nonreal poles, the order of growth of G near $\overline{\mathbf{R}}$ is finite (see Section 2.1) and there is a finite (possibly empty) subset e of $\overline{\mathbf{R}}$ such that every connected component of $\overline{\mathbf{R}} \setminus e$ is of definite type with respect to G (see Definition 2.5). We can use this characterization of definitizability of operator functions to introduce a local variant of this notion ([6, Definition 4.1], see Definition 2.9 below), that is, we define, in a natural way, operator functions definitizable in some domain Ω. In the same way as for definitizable operator functions open subsets of $\overline{\mathbf{R}}$ of type π_+ and π_- can be defined, which gives a localization of the characteristic properties of the generalized Nevanlinna functions (Section 2.3).

Let, in the following, Ω be a domain in $\overline{\mathbf{C}}$ which is symmetric with respect to $\mathbf{R}$, such that $\Omega \cap \overline{\mathbf{R}} \neq \emptyset$, and $\Omega \cap \mathbf{C}^+$ and $\Omega \cap \mathbf{C}^-$ are simply connected. Here $\mathbf{C}^+$ and $\mathbf{C}^-$ denote the open upper and the open lower half planes, respectively. An operator function G is definitizable in Ω if and only if for every domain Ω' with the same properties as Ω, and with $\overline{\Omega}' \subset \Omega$, the restriction of G to Ω' can be written as a sum of a definitizable operator function and an operator function holomorphic in Ω' (see Proposition 2.10).

The main objective of the present paper are representations of operator functions definitizable in Ω with the help of selfadjoint operators or selfadjoint relations definitizable in Ω (Section 3). We consider representations of the form studied in [3] for generalized Nevanlinna functions and in [5] for definitizable functions. A local variant of the notion of minimality of a representation is introduced (Definition 3.2). If a representation of an operator function G is locally minimal, then the local "sign properties" of G (including multiplicities) are exactly reflected by the local "sign properties" of the representing relation. Moreover, if A_1 and A_2 are two locally minimal locally definitizable representing relations for G, results from [5] on the "local unitary equivalence" of A_1 and A_2 for the case of a definitizable G remain true in our more general situation.

In Section 3.2 we shall show that for every domain Ω' with the same properties as Ω, and with $\overline{\Omega'} \subset \Omega$, there exists a locally minimal representation of the restriction of G to Ω' with the help of some selfadjoint relation A in a Krein space which is definitizable over Ω'. This will be proved with the help of a variant of T. Ya. Azizov's theorem on the representation of operator functions (Theorem 3.7): there exists a minimal representing selfadjoint relation with spectrum outside of an arbitrarily chosen compact subset of the set of holomorphy of the operator function.

By a linear fractional transformation of the independent variable and by making use of the corresponding Cayley transformation all definitions and results mentioned above can be carried over to similar definitions and equivalent results for operator functions skew-symmetric with respect to the unit circle $\mathbf{T}$. It is often convenient to give the proofs in the $\mathbf{T}$-skew-symmetric situation. Therefore, we shall formulate all definitions and most of the results for both situations.

2. Locally definitizable operator functions

2.1. Preliminaries on R-symmetric and T-skew-symmetric operator functions

For every subset M of $\overline{\mathbf{C}}$ we set $M^* := \{\bar{\lambda} : \lambda \in M\}$ and $\hat{M} := \{\bar{\lambda}^{-1} : \lambda \in M\}$. For a scalar function f defined on a set $M \subset \overline{\mathbf{C}}$ with $M = M^*$ ($M = \hat{M}$) we set $f^*(\lambda) := \overline{f(\bar{\lambda})}$ (resp. $\hat{f}(\lambda) := \overline{f(\bar{\lambda}^{-1})}$). If the values of f are bounded linear operators in a Krein space $\mathcal{H}$ we set $f^*(\lambda) := f(\overline{\lambda})^+$ (resp. $\hat{f}(\lambda) := f(\bar{\lambda}^{-1})^+$).

Let, in this and the following sections, Ω be a domain in $\overline{\mathbf{C}}$ with the properties mentioned in the introduction. Let $\lambda_0 \in \Omega \cap \mathbf{C}^+$,

$$\psi(\lambda) := -(\lambda - \lambda_0)(\lambda - \overline{\lambda_0})^{-1} \quad \phi(z) := (\overline{\lambda_0} z + \lambda_0)(z+1)^{-1}.$$

Then $\phi \circ \psi = \mathrm{id}$ and $\psi(\overline{\mathbf{R}}) = \mathbf{T}$. The domain $\psi(\Omega)$ is symmetric with respect to $\mathbf{T}$, $\psi(\Omega) \cap \mathbf{T}$ is not empty, $0, \infty \in \psi(\Omega)$, and $\psi(\Omega) \cap \mathbf{D}$ and $\psi(\Omega) \cap \hat{\mathbf{D}}$ are simply connected domains of $\overline{\mathbf{C}}$. Here $\mathbf{D}$ denotes the open unit disc.

Let G be an $\mathcal{L}(\mathcal{H})$-valued meromorphic function in $\Omega \setminus \overline{\mathbf{R}}$, $G = G^*$, such that no point of $\Omega \cap \overline{\mathbf{R}}$ is an accumulation point of nonreal poles of G. Let μ be a point of $\Omega \cap \overline{\mathbf{R}}$ such that G can be continued analytically in μ from $\Omega \cap \mathbf{C}^+$ and (hence, also) from $\Omega \cap \mathbf{C}^-$ and these analytic continuations coincide. In the following we will tacitly assume that G is defined also in these points μ, and by a "point of holomorphy" of G we will understand either a point of holomorphy of G in $\Omega \setminus \overline{\mathbf{R}}$ or a point $\mu \in \Omega \cap \overline{\mathbf{R}}$ with the property just mentioned.

If M is a closed subset of $\overline{\mathbf{C}}$ and $\mathcal{X}$ is a Banach space, the linear space of all locally holomorphic functions on M with values in $\mathcal{X}$ equipped with the usual topology (see [7, Section 27.4]) will be denoted by $H(M, \mathcal{X})$. We set $H(M) := H(M, \mathbf{C})$.

Assume that $\lambda_0 \in \Omega \cap \mathbf{C}^+$ is a point of holomorphy of G. Let $\mathcal{O}^+$ be a bounded C^∞-domain (not necessarily simply connected) with $\overline{\mathcal{O}^+} \subset \Omega \cap \mathbf{C}^+$ and $\lambda_0 \in \mathcal{O}^+$ such that G is locally holomorphic on $\overline{\mathcal{O}^+}$. Then by $G = G^*$, G is also locally holomorphic on $\overline{\mathcal{O}^-}$, $\mathcal{O}^- := (\mathcal{O}^+)^*$. For every $g \in H(\overline{\mathbf{C}} \setminus (\mathcal{O}^+ \cup \mathcal{O}^-))$ we define

$$S_G.g := -2i\,(\operatorname{Im}\lambda_0) \int_{\mathcal{C}} G(\lambda) g(\lambda)(\lambda - \lambda_0)^{-1}(\lambda - \overline{\lambda_0})^{-1}\, d\lambda, \tag{2.1}$$

where $\mathcal{C} = \partial\mathcal{O}^+ \cup \partial\mathcal{O}^-$ †. Evidently, for every function g locally holomorphic on

$$(\overline{\mathbf{C}} \setminus \Omega) \cup \overline{\mathbf{R}} \cup \{\text{poles of } G\}$$

we may find some domain $\mathcal{O}^+$ as above and such that $g \in H(\overline{\mathbf{C}} \setminus (\mathcal{O}^+ \cup \mathcal{O}^-))$. Then the operator $S_G.g$ is defined, and it does not depend on the choice of $\mathcal{O}^+$. S_G is a continuous linear mapping of $H((\overline{\mathbf{C}} \setminus \Omega) \cup \overline{\mathbf{R}} \cup \{\text{poles of } G\})$ into $\mathcal{L}(\mathcal{H})$. It is easy to see that $S_G.g^* = (S_G.g)^+$.

It is not difficult to find some right inverse of the mapping $G \mapsto S_G$: Let $\sigma_0 = \sigma_0^*$ be a countable subset of $\Omega \setminus \overline{\mathbf{R}}$ which has no accumulation points in Ω, $\lambda_0 \notin \sigma_0$, and let S be a continuous linear mapping of $H((\overline{\mathbf{C}} \setminus \Omega) \cup \overline{\mathbf{R}} \cup \sigma_0)$ in $\mathcal{L}(\mathcal{H})$ such that

$$S \,.\, g^* = (S \,.\, g)^* \quad \text{for all} \quad g \in H((\overline{\mathbf{C}} \setminus \Omega) \cup \overline{\mathbf{R}} \cup \sigma_0)$$

(or, equivalently, $S.g$ is selfadjoint for $g = g^* \in H((\overline{\mathbf{C}} \setminus \Omega) \cup \overline{\mathbf{R}} \cup \sigma_0)$) and S is of finite order at every point μ_0 of σ_0. That is, the restriction of S to the subspace of all functions $g \in H((\overline{\mathbf{C}} \setminus \Omega) \cup \overline{\mathbf{R}} \cup \sigma_0)$ which are zero in some neighbourhood of $((\overline{\mathbf{C}} \setminus \Omega) \cup \overline{\mathbf{R}} \cup \sigma_0) \setminus \{\mu_0\}$ has the form $g \longmapsto \sum_{\nu=0}^{k} A_\nu g^{(\nu)}(\mu_0)$ where $A_\nu \in \mathcal{L}(\mathcal{H})$, $\nu = 0, \cdots, k$, for some $k \in \mathbf{N}$. We denote the linear space of these mappings by $\Phi(\Omega, \overline{\mathbf{R}} \cup \sigma_0; \mathcal{L}(\mathcal{H}))$. If G is as above then S_G belongs to this space where σ_0 is the set of poles of G in $\Omega \setminus \overline{\mathbf{R}}$. For $S \in \Phi(\Omega, \overline{\mathbf{R}} \cup \sigma_0; \mathcal{L}(\mathcal{H}))$ we define

$$G_S(\lambda) := S \,.\, g_\lambda \quad \text{where}$$

$$g_\lambda(w) := (4\pi)^{-1}(\operatorname{Im} \lambda_0)^{-1}(\lambda - \operatorname{Re} \lambda_0 + (\lambda - \lambda_0)(\lambda - \overline{\lambda_0})(w - \lambda)^{-1}).$$

The function G_S fulfils the general assumptions on the operator functions G considered in this section. It is not difficult to verify that

$$S_{G_S} = S.$$

If G is as at the beginning of this section, then

$$G(\lambda) - \frac{1}{2}(G(\lambda_0) + G(\lambda_0)^+) = S_G \,.\, g_\lambda (= G_{S_G}(\lambda))$$

for all points λ of holomorphy of G in $\Omega \setminus \mathbf{R}$.

As in [5, Section 3] besides the operator-valued functional S_G we consider a form-valued functional $S_G(\cdot,\cdot)$. Let $\mathcal{O}^+$, $\mathcal{O}^-$, $\mathcal{C}$ and g be as in the definition of S_G, and let u, v be $\mathcal{H}$-valued functions locally holomorphic on $\overline{\mathbf{C}} \setminus (\mathcal{O}^+ \cup \mathcal{O}^-)$, that is $u, v \in H(\overline{\mathbf{C}} \setminus (\mathcal{O}^+ \cup \mathcal{O}^-), \mathcal{H})$. Then we set

$$S_G(u,v).g := -2i\,(\operatorname{Im} \lambda_0) \int_{\mathcal{C}} [G(\lambda)u(\lambda), v(\overline{\lambda})] g(\lambda)(\lambda - \lambda_0)^{-1}(\lambda - \overline{\lambda_0})^{-1}\, d\lambda. \quad (2.2)$$

This defines $S_G(\cdot,\cdot).(\cdot)$ for all ($\mathcal{H}$-valued and scalar, respectively) functions locally holomorphic on $H((\overline{\mathbf{C}} \setminus \Omega) \cup \overline{\mathbf{R}} \cup \{\text{poles of } G\})$. If $g = g^*$, the sesquilinear form $(u, v) \mapsto S_G(u, v).g$ is hermitian.

Let F be a $\mathcal{L}(\mathcal{H})$-valued meromorphic function in $\psi(\Omega \setminus \overline{\mathbf{R}}) = \psi(\Omega) \setminus \mathbf{T}$ which is skew-symmetric with respect to the unit circle $\mathbf{T}$: $\hat{F} = -F$. Assume that no

† In the definition of S_G in [6], relation (3.8), a minus sign is missing.

point of $\psi(\Omega)\cap\mathbf{T}$ is an accumulation point of non-unimodular poles of F and that F is holomorphic at 0 and ∞. Then

$$G := iF \circ \psi \tag{2.3}$$

satisfies the assumptions mentioned at the beginning of this section. If $\mathcal{O}^+$ and $\mathcal{O}^-$ are as above then for every $f \in H(\overline{\mathbf{C}} \setminus \psi(\mathcal{O}^+ \cup \mathcal{O}^-))$ we define

$$T_F.\, f := \int_{\psi(\mathcal{C})} F(z)f(z)(iz)^{-1}dz, \tag{2.4}$$

where $\psi(\mathcal{C}) = \partial\psi(\mathcal{O}^+) \cup \partial\psi(\mathcal{O}^-)$. Similarly to the definition of S_G, in this way the operator $T_F.f$ is defined for every function f which is locally holomorphic on $(\overline{\mathbf{C}} \setminus \psi(\Omega)) \cup \mathbf{T} \cup \{\text{poles of } F\}$.

Below we will make use of some right inverse of the mapping $F \mapsto T_F$: Let $\tau_0 = \hat{\tau}_0$ be a countable bounded subset of $\psi(\Omega) \setminus \mathbf{T}$ which has no accumulation points in Ω, and let T be a continuous linear mapping of $H((\overline{\mathbf{C}} \setminus \psi(\Omega)) \cup \mathbf{T} \cup \tau_0)$ in $\mathcal{L}(\mathcal{H})$ such that

$$T\,.\,\hat{f} = (T\,.\,f)^+ \quad \text{for all} \quad f \in H((\overline{\mathbf{C}} \setminus \psi(\Omega)) \cup \mathbf{T} \cup \tau_0)$$

(or, equivalently, $T.f$ is selfadjoint for all $f = \hat{f} \in H((\overline{\mathbf{C}} \setminus \psi(\Omega)) \cup \mathbf{T} \cup \tau_0))$ and T is of finite order at every point of τ_0. The linear space of these mappings is denoted by $\Phi(\psi(\Omega), \mathbf{T} \cup \tau_0; \mathcal{L}(\mathcal{H}))$. If F is as above then T_F belongs to this space where τ_0 is the set of all poles of F in $\psi(\Omega) \setminus \mathbf{T}$. If $T \in \Phi(\psi(\Omega), \mathbf{T} \cup \tau_0; \mathcal{L}(\mathcal{H}))$ and $\zeta \in \psi(\Omega) \setminus (\mathbf{T} \cup \tau_0)$ we define an operator function F_T by

$$F_T(\zeta) := T\,.\,h_\zeta \quad \text{where} \quad h_\zeta(z) := (4\pi)^{-1}(z+\zeta)(z-\zeta)^{-1}.$$

Then F_T is meromorphic in $\psi(\Omega) \setminus \mathbf{T}$, the poles of F_T in $\psi(\Omega) \setminus \mathbf{T}$ are contained in τ_0 and we have $\widehat{F_T} = -F_T$. Moreover,

$$T_{F_T} = T. \tag{2.5}$$

If F is as above then

$$F(z) - \frac{1}{2}(F(0) - F(0)^+) = T_F.h_z (= F_{T_F}(z)) \tag{2.6}$$

for all points of holomorphy of F in $\psi(\Omega) \setminus \mathbf{T}$.

If, again, the operator function F is as above, $f \in H(\overline{\mathbf{C}} \setminus \psi(\mathcal{O}^+ \cup \mathcal{O}^-))$ and $p, q \in H(\overline{\mathbf{C}} \setminus \psi(\mathcal{O}^+ \cup \mathcal{O}^-), \mathcal{H})$, we define

$$T_F(p,q).f := \int_{\psi(\mathcal{C})} [F(z)p(z), q(\bar{z}^{-1})] f(z)(iz)^{-1}dz.$$

If (2.3) holds then for all functions f, p, q which are locally holomorphic on $(\overline{\mathbf{C}} \setminus \psi(\Omega)) \cup \mathbf{T} \cup \{\text{poles of } F\}$ (scalar and with values in $\mathcal{H}$, respectively) we have

$$T_F.f = S_G.(f \circ \psi), \tag{2.7}$$

$$T_F(u \circ \phi, v \circ \phi).(g \circ \phi) = S_G(u,v).g. \tag{2.8}$$

Let Δ be an open subset of $\Omega \cap \overline{\mathbf{R}}$, and let $m \geq 1$. We shall say that the *order of growth of* G *near* Δ *is* $\leq m$, if for every closed subset Δ' of Δ there exists a constant M and an open neighbourhood $\mathcal{U}$ of Δ' in $\overline{\mathbf{C}}$ such that

$$\|G(\lambda)\| \leq M(1+|\lambda|)^{2m}|\mathrm{Im}\,\lambda|^{-m}$$

for all $\lambda \in \mathcal{U} \setminus \overline{\mathbf{R}}$. We do not exclude the case when $\Omega = \overline{\mathbf{C}}$ and $\Delta = \overline{\mathbf{R}}$.

Analogously, if Γ is an open subset of $\psi(\Omega) \cap \mathbf{T}$ we shall say that the *order of growth of* F *near* Γ *is* $\leq m$, if for every closed subset Γ' of Γ there exists a constant M and an $r_0 \in (0,1)$ such that

$$\|F(re^{i\Theta})\| \leq M|1-|r||^{-m}$$

for all $e^{i\Theta} \in \Gamma'$ and $r \in [r_0, 1) \cup (1, r_0^{-1}]$.

2.2. Extension of the functionals associated with G and F and its consequences

Assume that the order of growth of G near $\Delta \subset \Omega \cap \overline{\mathbf{R}}$ is $\leq m$. It is easy to verify that this is equivalent to the fact that the order of growth of F near $\psi(\Delta)$ is $\leq m$.

Let Γ_0 be the union of a finite number of pairwise disjoint open arcs of $\mathbf{T}$, $\Gamma_0 \neq \mathbf{T}$, and let $\delta_0 \in (0,1)$ be such that for

$$Q_0 := \{re^{i\Theta} : e^{i\Theta} \in \Gamma_0,\, r \in (\delta_0, 1) \cup (1, \delta_0^{-1})\} \tag{2.9}$$

the function F is locally holomorphic on $\overline{Q_0} \setminus \overline{\Gamma_0}$.

We denote by $D^{(p)}(\overline{\mathbf{C}} \setminus Q_0, \mathcal{H})$, p nonnegative integer, the linear space of all continuous $\mathcal{H}$-valued functions f on $\overline{\mathbf{C}} \setminus Q_0$ such that f is locally holomorphic on $\overline{\mathbf{C}} \setminus (Q_0 \cup \Gamma_0)$ and the restriction $f|\mathbf{T}$ is a C^p function. For $D^{(p)}(\overline{\mathbf{C}} \setminus Q_0, \mathbf{C})$ we simply write $D^{(p)}(\overline{\mathbf{C}} \setminus Q_0)$. We introduce a locally convex topology on $D^{(p)}(\overline{\mathbf{C}} \setminus Q_0, \mathcal{H})$: Let ϵ_0, $0 < \epsilon_0 < 1 - \delta_0$, be such that for $0 < \epsilon < \epsilon_0$ every component of Γ_0 contains a point of

$$\Gamma_\epsilon := \{e^{i\Theta} \in \Gamma_0 : \mathrm{dist}\,(e^{i\Theta}, \mathbf{T} \setminus \Gamma_0) > \epsilon\} \subset \Gamma_0,$$

and set

$$Q_\epsilon := \{re^{i\Theta} : e^{i\Theta} \in \Gamma_\epsilon,\, r \in (\delta_0 + \epsilon, 1) \cup (1, (\delta_0 + \epsilon)^{-1})\}.$$

Let $(\epsilon_n) \subset (0, \epsilon_0)$ be a decreasing null sequence and let $D_n^{(p)}$ be the subspace of $D^{(p)}(\overline{\mathbf{C}} \setminus Q_0, \mathcal{H})$ of all $f \in D^{(p)}(\overline{\mathbf{C}} \setminus Q_0, \mathcal{H})$ which can analytically be continued to $\overline{\mathbf{C}} \setminus \overline{(Q_{\epsilon_n} \cup \Gamma_{\epsilon_n})}$ such that f is continuous on $\overline{\mathbf{C}} \setminus (Q_{\epsilon_n} \cup \Gamma_{\epsilon_n})$. Evidently, we have $D^{(p)}(\overline{\mathbf{C}} \setminus Q_0, \mathcal{H}) = \bigcup_{n=1}^\infty D_n^{(p)}$. On the space $D_n^{(p)}$ we consider the norm

$$\begin{aligned} \|f\|_n^{(p)} := \sup &\left\{\|f(z)\| : z \in \overline{\mathbf{C}} \setminus \overline{(Q_{\epsilon_n} \cup \Gamma_{\epsilon_n})}\right\} + \\ &+ \sup\left\{\left\|\tfrac{d^\nu}{d\Theta^\nu} f(e^{i\Theta})\right\| : e^{i\Theta} \in \Gamma_0,\, 0 \leq \nu \leq p\right\},\ f \in D_n^{(p)}. \end{aligned}$$

$(D_n^{(p)}, \|f\|_n^{(p)})$ is a Banach space. On the space $D^{(p)}(\overline{\mathbf{C}} \setminus Q_0, \mathcal{H})$ we consider the topology of the inductive limit of the spaces $D_n^{(p)}$, $n = 1, 2, \ldots$. One verifies as in [7, §27,4.(2)] that this topology is separated. By well-known properties of the Abel-Poisson integral, $H(\overline{\mathbf{C}} \setminus Q_0)$ is dense in $D^{(p)}(\overline{\mathbf{C}} \setminus Q_0)$.

It was proved in [6, Theorem 3.1] that for $\overline{\Gamma}_0 \subset \psi(\Delta)$ and under the above growth assumption on F, T_F is continuous on $H(\overline{\mathbf{C}} \setminus Q_0)$ with respect to the topology of $D^{(m+1)}(\overline{\mathbf{C}} \setminus Q_0)$. Therefore T_F can be extended by continuity to $D^{(m+1)}(\overline{\mathbf{C}} \setminus Q_0)$. By (2.7) S_G is continuous on $H(\overline{\mathbf{C}} \setminus \phi(Q_0))$ with respect to the topology in $H(\overline{\mathbf{C}} \setminus \phi(Q_0))$ induced by the topology defined above and the mapping

$$H(\overline{\mathbf{C}} \setminus Q_0) \ni f \longmapsto f \circ \psi \in H(\overline{\mathbf{C}} \setminus \phi(Q_0)).$$

We extend S_G to all functions defined on $\overline{\mathbf{C}} \setminus \phi(Q_0)$ and belonging to the space $D^{(m+1)}(\overline{\mathbf{C}} \setminus Q_0) \circ \psi = \{f \circ \psi : \ f \in D^{(m+1)}(\overline{\mathbf{C}} \setminus Q_0)\}$:

$$S_G.(f \circ \phi) := T_F.f, \qquad f \in D^{(m+1)}(\overline{\mathbf{C}} \setminus Q_0).$$

In particular, the extended functionals T_F and S_G are defined on all functions $f \in D^{(m+1)}(\overline{\mathbf{C}} \setminus Q_0)$ and $g \in D^{(m+1)}(\overline{\mathbf{C}} \setminus Q_0) \circ \psi$, respectively, such that f and g are zero outside compact subsets of $\psi(\Delta)$ and Δ, respectively. In these cases, for brevity, we shall write $f \in C_0^{m+1}(\psi(\Delta))$ and $g \in C_0^{m+1}(\Delta)$. If we regard $\overline{\mathbf{R}}$ as a real-analytic manifold in the usual way, then the restriction of ψ to $\overline{\mathbf{R}}$ is a real-analytic diffeomorphism of $\overline{\mathbf{R}}$ onto $\mathbf{T}$, and therefore the linear space of the restrictions of the functions of $C_0^{m+1}(\Delta)$ to $\overline{\mathbf{R}}$ coincides with the linear space of the C^{m+1}-functions g on $\overline{\mathbf{R}}$ with supp $g \in \Delta$. If $f \in C_0^{m+1}(\psi(\Delta))$ is a real function, then it can be approximated in $D^{(m+1)}(\overline{\mathbf{C}} \setminus Q_0)$ by a sequence of functions $f_n \in H(\overline{\mathbf{C}} \setminus Q_0)$ with $f_n = \hat{f}_n$, hence, $T_F.f$ is selfadjoint. Similarly, $S_G.g$ is selfadjoint for real functions $g \in C_0^{m+1}(\Delta)$.

We will make use of the following proposition (cf. [4, Section 1.3]).

Proposition 2.1. *Assume that the order of growth of G near to the open subset Δ of $\Omega \cap \overline{\mathbf{R}}$ is $\leq m$, and let Δ_0 be the union of a finite number of pairwise disjoint connected open subsets of Δ such that $\overline{\Delta_0} \subset \Delta$. Then the following holds.*

(i) *G can be written as a sum*

$$G = G_0 + G_{(0)},$$

where G_0 and $G_{(0)}$ are $\mathcal{L}(\mathcal{H})$-valued meromorphic functions in $\Omega \setminus \overline{\mathbf{R}}$, $G_0 = G_0^*$ is locally holomorphic on $\overline{\mathbf{C}} \setminus \Delta$, has growth of order $\leq m+2$ near $\overline{\mathbf{R}}$, and $G_{(0)} = G_{(0)}^*$ is locally holomorphic on $\overline{\Delta_0}$.

(ii) *F can be written as a sum*

$$F = F_0 + F_{(0)},$$

where F_0 and $F_{(0)}$ are $\mathcal{L}(\mathcal{H})$-valued meromorphic functions in $\psi(\Omega) \setminus \mathbf{T}$ such that $F_0 = -\hat{F}_0$ is locally holomorphic on $\overline{\mathbf{C}} \setminus \psi(\Delta)$, has growth of order $\leq m+2$ near $\mathbf{T}$, and $F_{(0)} = -\hat{F}_{(0)}$ is locally holomorphic on $\overline{\psi(\Delta_0)}$.

Proof. It is sufficient to prove assertion (ii). For every point z of holomorphy of F we have (see (2.6))

$$F(z) = T_F.h_z + \frac{1}{2}(F(0) - F(0)^+).$$

Let $\alpha \in C_0^{m+1}(\psi(\Delta))$ be real on $\mathbf{T}$ and equal to 1 on some neighbourhood of $\overline{\psi(\Delta_0)}$. We set

$$F_0(z) := T_F.\alpha h_z + \frac{1}{2}(F(0) - F(0)^+) \tag{2.10}$$

Let τ_0 denote the set of all poles of F in $\psi(\Omega)\setminus\mathbf{T}$. The operator $T_F.\alpha f$ is selfadjoint for every $f = \hat{f} \in H((\overline{\mathbf{C}}\setminus\psi(\Omega))\cup\mathbf{T}\cup\tau_0)$. Then it is easy to see that the functional

$$\alpha T_F : \quad f \longmapsto T_F.\alpha f$$

belongs to $\Phi(\psi(\Omega), \mathbf{T}\cup\tau_0; \mathcal{L}(\mathcal{H}))$. Therefore, $\hat{F}_0 = -F_0$. By the continuity properties of T_F, F_0 is complex differentiable outside of the support of α and, hence, locally holomorphic on $\overline{\mathbf{C}}\setminus\psi(\Delta)$. We define

$$F_{(0)}(z) := T_F.(1-\alpha)h_z.$$

Then $F = F_0 + F_{(0)}$. Since $1-\alpha$ is zero in some neighbourhood of $\overline{\psi(\Delta_0)}$ we conclude that $F_{(0)}$ is complex differentiable in some neighbourhood (in $\overline{\mathbf{C}}$) of any point of $\overline{\psi(\Delta_0)}$.

Let K be a compact subset of $\mathbf{C}\setminus\{0\}$. Then by the definition of F_0 and the local C^{m+1}-continuity of T_F there exist constants M and M' such that $z \in K\setminus\mathbf{T}$ implies

$$\begin{aligned}\|F_0(z)\| &\le M \sup\left\{\left|\frac{d^k}{d\Theta^k}h_z(e^{i\Theta})\right| :\ \Theta\in[0,2\pi],\ k = 0,\dots,m+1\right\}\\ &\le M'|1-|z||^{m+2}.\end{aligned}$$

That is, the growth of F_0 near $\mathbf{T}$ is of order $\le m+2$. □

Lemma 2.2. *Assume that G, G_0, F and F_0 are as in Proposition 2.1. Then*

$$S_G.g = S_{G_0}.g \quad \textit{for all} \quad g \in C_0^{m+3}(\Delta_0), \tag{2.11}$$

$$T_F.f = T_{F_0}.f \quad \textit{for all} \quad f \in C_0^{m+3}(\psi(\Delta_0)). \tag{2.12}$$

Proof. If $\Gamma_0 := \psi(\Delta_0)$ and Q_0 is as in (2.9), then every $f \in C_0^{m+3}(\Gamma_0)$ can be approximated in $D_n^{(m+3)}$ for some n by a sequence (f_k) of functions belonging to $H(\overline{\mathbf{C}}\setminus Q_0)$. Then, if $F_{(0)}$ is as in Proposition 2.1, by the definition of $T_{F_{(0)}}$, $T_{F_{(0)}}.f = \lim_{k\to\infty} T_{F_{(0)}}.f_k = 0$, which implies the lemma. □

Local growth properties of G and F imply also local continuity properties of the functionals $S_G(\cdot,\cdot)$ and $T_F(\cdot,\cdot)$ similar to those of S_G and T_F.

Proposition 2.3. *Assume that the order of growth of F near to the open subset Γ of $\psi(\Omega)\cap\mathbf{T}$ is $\le m$. Let Γ_0 be the union of a finite number of pairwise disjoint open subarcs of Γ such that $\overline{\Gamma}_0 \subset \Gamma$ and let Q_0 be as in (2.9). Then*

$$H(\overline{\mathbf{C}}\setminus Q_0, \mathcal{H})^2 \times H(\overline{\mathbf{C}}\setminus Q_0) \ni (p,q,f) \longmapsto T_F(p,q).f$$

is continuous with respect to the topology of $(D^{(m+3)}(\overline{\mathbf{C}}\setminus Q_0,\mathcal{H}))^2 \times D^{(m+3)}(\overline{\mathbf{C}}\setminus Q_0)$.

Proof. Let O_1^+ be a simply connected C^∞-subdomain of $\mathbf{D}$ with the following properties.

(i) $0 \in O_1^+$, $Q_0 \cap \mathbf{D} \subset O_1^+$, $\overline{O_1^+} \cap \mathbf{T} = \overline{\Gamma}_0$.

(ii) F is holomorphic in O_1^+ and in all points of $\overline{O_1^+} \setminus \mathbf{T}$.

Then there exists an $r_0 \in (0,1)$ such that for all $r \in [r_0, 1)$, F is holomorphic on the closure of $rO_1^+ := \{rz :\ z \in O_1^+\}$. We define $\mathcal{O}_r := rO_1^+ \cup (rO_1^+)\hat{}\,$, $r \in [r_0, 1]$.

Let F_0 and $F_{(0)}$ be as in Proposition 2.1, (ii), with $\psi(\Delta) = \Gamma$ and $\psi(\Delta_0) = \Gamma_0$. If $p, q \in H(\overline{\mathbf{C}} \setminus Q_0, \mathcal{H})$, $f \in H(\overline{\mathbf{C}} \setminus Q_0)$, then for sufficiently small $1 - r > 0$ we have

$$\begin{aligned} T_F(p,q).f &= \int_{\partial \mathcal{O}_r} [F(z)p(z), q(\bar{z}^{-1})] f(z)(iz)^{-1} dz \\ &= \int_{\partial \mathcal{O}_r} [F_0(z)p(z), q(\bar{z}^{-1})] f(z)(iz)^{-1} dz + \\ &\quad + \int_{\partial \mathcal{O}_r} [F_{(0)}(z)p(z), q(\bar{z}^{-1})] f(z)(iz)^{-1} dz \\ &= \int_{\partial(r\mathbf{D} \cup r^{-1}\hat{\mathbf{D}})} [F_0(z)p(z), q(\bar{z}^{-1})] f(z)(iz)^{-1} dz + \\ &\quad + \int_{\partial(\mathcal{O}_1 \setminus \bar{\Gamma}_0)} [F_{(0)}(z)p(z), q(\bar{z}^{-1})] f(z)(iz)^{-1} dz. \end{aligned} \tag{2.13}$$

F_0 has growth of order $\leq m+2$ near $\mathbf{T}$. Then by [4, Proposition 1.2] the first term on the right hand side of (2.13) is continuous on $(C^{m+3}(\mathbf{T}, \mathcal{H}))^2 \times C^{m+3}(\mathbf{T})$. Since the topologies of $D^{(m+3)}(\overline{\mathbf{C}} \setminus Q_0, \mathcal{H})$ and $D^{(m+3)}(\overline{\mathbf{C}} \setminus Q_0)$ are stronger than those of $(C^{m+3}(\mathbf{T}, \mathcal{H}))$ and $C^{m+3}(\mathbf{T})$, respectively, the first term on the right hand side of (2.13) is continuous with respect to the topology mentioned in the proposition. As to the second term on the right hand side of (2.13), there is a constant M such that the absolute value of the second term can be estimated from above by

$$M \ \sup\{\|p(z)\| :\ z \in \overline{\mathbf{C}} \setminus \bar{Q}_0\} \ \sup\{\|q(z)\| :\ z \in \overline{\mathbf{C}} \setminus \bar{Q}_0\} \times \sup\{|f(z)| :\ z \in \overline{\mathbf{C}} \setminus \bar{Q}_0\}.$$

This implies Proposition 2.3.

□

If the assumptions of Proposition 2.3 are fulfilled, by (2.8) a similar continuity statement holds for $S_G(\cdot,\cdot)$ and the topologies induced by the mapping $f \longmapsto f \circ \psi$. For the extended functional $S_G(\cdot,\cdot)$ we have

$$S_G(p \circ \phi, q \circ \phi).(f \circ \phi) := T_F(p,q).f, \quad p, q \in D^{(m+3)}(\overline{\mathbf{C}} \setminus Q_0, \mathcal{H}), \quad f \in D^{(m+3)}(\overline{\mathbf{C}} \setminus Q_0). \tag{2.14}$$

In the same way as in Lemma 2.2 and making use of (2.14) we verify the following.

Lemma 2.4. *If F, F_0, G and G_0 are as in Proposition* 2.1*, then*

$$T_F(p,q).f = T_{F_0}(p,q).f$$

for all $p, q \in C_0^{m+3}(\psi(\Delta_0), \mathcal{H})$, $f \in C_0^{m+3}(\psi(\Delta_0))$, and

$$S_G(u,v).g = S_{G_0}(u,v).g$$

for all $u, v \in C_0^{m+3}(\Delta_0, \mathcal{H})$, $g \in C_0^{m+3}(\Delta_0)$.

2.3. Open sets of positive and negative type with respect to an operator function

Let G and F be as in Sections 2.1 and 2.2. The following definitions of open sets of positive and negative type with respect to the operator functions G and F are equivalent to those in [6, Definitions 3.7 and 3.9]. For these and further equivalent descriptions of these sets see [6, Lemma 3.8 and Lemma 3.10].

Definition 2.5. An open subset $\Delta \subset \Omega \cap \overline{\mathbf{R}}$ is said to be of *positive type with respect to G* if for every $x \in \mathcal{H}$ and every sequence (λ_n) of points of holomorphy of G in $\Omega \cap \mathbf{C}^+$ which converges in $\overline{\mathbf{C}}$ to a point of Δ we have

$$\liminf_{n\to\infty} \mathrm{Im}\, [G(\lambda_n)x, x] \geq 0.$$

An open subset $\Delta \subset \Omega \cap \overline{\mathbf{R}}$ is said to be of *negative type with respect to G* if Δ is of positive type with respect to $-G$. Δ is said to be of *definite type with respect to G* if Δ is of positive type or of negative type with respect to G. A point $\lambda \in \Omega \cap \overline{\mathbf{R}}$ which is not contained in an open set of definite type with respect to G is called a *critical point of G in Ω*, we write $\lambda \in \mathrm{K}(G, \Omega)$.

Definition 2.5′**.** An open set $\Gamma \subset \psi(\Omega) \cap \mathbf{T}$ is said to be of *positive type with respect to F* if for every $x \in \mathcal{H}$ and every convergent sequence $(z_n) \subset \psi(\Omega) \cap \mathbf{D}$ of points of holomorphy of F with $\lim_{n\to\infty} z_n \in \Gamma$ we have

$$\liminf_{n\to\infty} \mathrm{Re}\, [F(z_n)x, x] \geq 0.$$

An open set $\Gamma \subset \psi(\Omega) \cap \mathbf{T}$ is said to be of *negative type with respect to F* if Γ is of positive type with respect to $-F$. Γ is said to be of *definite type with respect to F* if Γ is of positive type or of negative type with respect to F. A point $z \in \psi(\Omega) \cap \mathbf{T}$ which is not contained in an open set of definite type with respect to F is called a *critical point of F in $\psi(\Omega)$*, we write $z \in \mathrm{K}(F, \psi(\Omega))$.

Assume that $\Omega = \overline{\mathbf{C}}$ and let G, in addition, be piecewise holomorphic in $\mathbf{C} \setminus \mathbf{R}$. Then $\overline{\mathbf{R}}$ is of positive type with respect to G if and only if G is a Nevanlinna function, i.e. $\mathrm{Im}\, [G(\lambda)x, x] \geq 0$ for every $\lambda \in \mathbf{C}^+$ and every $x \in \mathcal{H}$. This is a consequence of the fact that a harmonic function does not attain its minimum in the interior of its domain. If $\Omega = \overline{\mathbf{C}}$ and F is piecewise holomorphic in $\overline{\mathbf{C}} \setminus \mathbf{T}$, then $\mathbf{T}$ is of positive type with respect to F if and only if F is a Caratheodory function, i.e. $\mathrm{Re}\, [F(z)x, x] \geq 0$ for every $z \in \mathbf{D}$ and every $x \in \mathcal{H}$.

Proposition 2.6. *Let Δ be an open subset of $\Omega \cap \overline{\mathbf{R}}$. Then the following statements are equivalent.*

(i) Δ *is of positive type with respect to* G.
(i′) $\psi(\Delta)$ *is of positive type with respect to* F.
(ii) *The order of growth of* G *near* Δ *is* $\leq m$ *for some positive integer* m, *and* $[(S_G.g)x, x] \geq 0$ *for every nonnegative function* $g \in C_0^\infty(\Delta)$ *and any* $x \in \mathcal{H}$.
(ii′) *The order of growth of* F *near* $\psi(\Delta)$ *is* $\leq m$ *for some integer* m, *and* $[(T_F.f)x, x] \geq 0$ *for every nonnegative function* $f \in C_0^\infty(\psi(\Delta))$ *and any* $x \in \mathcal{H}$.
(iii) *The order of growth of* G *near* Δ *is* $\leq m$ *for some positive integer* m, *and* $S_G(\cdot,\cdot).\mathbf{1}$ *restricted to* $C_0^\infty(\Delta, \mathcal{H})$ *is positive semidefinite.*
(iii′) *The order of growth of* F *near* $\psi(\Delta)$ *is* $\leq m$ *for some positive integer* m, *and* $T_F(\cdot,\cdot).\mathbf{1}$ *restricted to* $C_0^\infty(\psi(\Delta), \mathcal{H})$ *is positive semidefinite.*
(iv) *For every open set* Δ_0 *which is the union of a finite number of pairwise disjoint connected open subsets of* Δ *such that* $\overline{\Delta_0} \subset \Delta$, G *can be written as a sum* $G = G_0 + G_{(0)}$, *where* G_0 *is an* $\mathcal{L}(\mathcal{H})$*-valued Nevanlinna function and* $G_{(0)}$ *is locally holomorphic on* $\overline{\Delta_0}$.
(iv′) *For every open set* Γ_0 *which is the union of a finite number of pairwise disjoint connected open subsets of* $\psi(\Delta)$ *such that* $\overline{\Gamma}_0 \subset \psi(\Delta)$, F *can be written as a sum* $F = F_0 + F_{(0)}$, *where* F_0 *is an* $\mathcal{L}(\mathcal{H})$*-valued Caratheodory function and* $F_{(0)}$ *is locally holomorphic on* $\overline{\Gamma}_0$.

Proof. By relation (2.3) the statements (i) and (i′) are equivalent. Since (2.7) and (2.8) remain true for the extended functionals S_G, T_F, $S_G(\cdot,\cdot).\mathbf{1}$, $T_F(\cdot,\cdot).\mathbf{1}$, (ii) and (ii′) as well as (iii) and (iii′) are equivalent. On account of [6, Lemmas 3.10 and 3.12] (i′) and (ii′) are equivalent.

Assume that (i′) and (ii′) hold. We show that (iv′) holds. We construct a decomposition of F as in the proof of Proposition 2.1 with $\psi(\Delta_0) = \Gamma_0$ and assume, in addition, that the function α in (2.10) is nonnegative. It follows from (2.6) and (2.5) that

$$T_{F_0}.f = T_F.\,\alpha f \text{ for all } f \in C^\infty(\mathbf{T}).$$

This relation shows that, for every $x \in \mathcal{H}$, $[T_{F_0}.(\cdot)x, x]$ is a nonnegative functional on $C^\infty(\mathbf{T})$, i.e. F_0 is a Caratheodory function. It is easy to see that (iv′) implies (i′). (iv) and (iv′) are equivalent.

Assume that (iv′) holds. Let $u, v \in C_0^\infty(\Gamma_0, \mathcal{H})$ and let f_0 be a real function in $C_0^\infty(\Gamma_0)$ which is equal to one on the supports of u and v. Then, by Lemma 2.4,

$$T_F(u, v).\mathbf{1} = T_F(u, v).f_0^2 = T_{F_0}(u, v).f_0^2 = T_{F_0}(u, v).\mathbf{1}.$$

Since F_0 is a Caratheodory function the form $T_{F_0}(\cdot,\cdot).\mathbf{1}$ is positive semidefinite (see [4, Lemma 1.7]). Therefore, $T_F(\cdot,\cdot).\mathbf{1}$ is positive semidefinite on $C_0^\infty(\Gamma_0, \mathcal{H})$. This implies (iii′).

If (iii′) holds, then for every nonnegative $f_1 \in C_0^\infty(\psi(\Delta))$ and every $x \in \mathcal{H}$ we have

$$0 \leq T_F(f_1 x, f_1 x).\mathbf{1} = [(T_F.f_1^2)x, x].$$

Since every nonnegative function $f \in C_0^\infty(\psi(\Delta))$ restricted to $\mathbf{T}$ can be approximated in $C^k(\mathbf{T})$ for every $k = 0, 1, \ldots,$ by functions of the form f_1^2, $f_1 \in C_0^\infty(\psi(\Delta))$, we obtain (ii′), and Proposition 2.6 is proved. $\square$

2.4. Open sets of type π_+ and π_- with respect to an operator function

If $\mathcal{L}$ is a linear space equipped with a Hermitian sesquilinear form $[\cdot,\cdot]$, we denote by $\kappa_+((\mathcal{L},[\cdot,\cdot]))$ ($\kappa_-((\mathcal{L},[\cdot,\cdot]))$) the least upper bound ($\le \infty$) of the dimensions of $[\cdot,\cdot]$-positive definite (resp. $[\cdot,\cdot]$-negative definite) subspaces of $\mathcal{L}$. These quantities are called the *rank of positivity* and the *rank of negativity of* $[\cdot,\cdot]$ *on* $\mathcal{L}$.

Let G and F be as in Sections 2.1 and 2.2 and let the order of growth of G near to an open set $\Delta \subset \Omega \cap \overline{\mathbf{R}}$ be $\le m$ for some positive integer m. Then we define

$$\kappa_\pm(\Delta, G) := \kappa_\pm((C_0^\infty(\Delta,\mathcal{H}), S_G(\cdot,\cdot).\mathbf{1})).$$

Analogously, for an open set $\Gamma \subset \psi(\Omega) \cap \mathbf{T}$ we put

$$\kappa_\pm(\Gamma, F) := \kappa_\pm((C_0^\infty(\Gamma,\mathcal{H}), T_F(\cdot,\cdot).\mathbf{1})).$$

By Proposition 2.6, Δ (Γ) is of positive type with respect to G (resp. F) if and only if $\kappa_-(\Delta, G) = 0$ (resp. $\kappa_-(\Gamma, F) = 0$). Analogously for Δ and Γ of negative type.

Definition 2.7. An open set $\Delta \subset \Omega \cap \overline{\mathbf{R}}$ ($\Gamma \subset \psi(\Omega) \cap \mathbf{T}$) is said to be of *type* π_+ with respect to G (resp. F) if the order of growth of G (resp. F) near to Δ (resp. Γ) is finite and for every open subset Δ_0, $\overline{\Delta_0} \subset \Delta$, (resp. Γ_0, $\overline{\Gamma_0} \subset \Gamma$) we have $\kappa_-(\Delta_0, G) < \infty$ (resp. $\kappa_-(\Gamma_0, F) < \infty$).

Analogously, with κ_- replaced by κ_+, sets of *type* π_- with respect to G and F are defined.

Assume that $\Omega = \overline{\mathbf{C}}$. Then the set of all piecewise meromorphic $\mathcal{L}(\mathcal{H})$-valued functions $G_0 = G_0^*$ in $\mathbf{C} \setminus \mathbf{R}$ such that the total multiplicity of the poles of G_0 in $\mathbf{C}^+$ is finite, the growth of G_0 near $\overline{\mathbf{R}}$ is of finite order and $\kappa_-(\overline{\mathbf{R}}, G_0) < \infty$ holds, coincides with the set of all generalized Nevanlinna functions, i.e. with the union of all Krein-Langer classes $N_k(\mathcal{L}(\mathcal{H}))$, $k = 0, 1, \ldots$. Similarly, the set of all piecewise meromorphic $\mathcal{L}(\mathcal{H})$-valued functions $F_0 = -\hat{F}_0$ in $\overline{\mathbf{C}} \setminus \mathbf{T}$ such that the total multiplicity of the poles of F_0 in $\mathbf{D}$ is finite, the growth of F_0 near $\mathbf{T}$ is of finite order and $\kappa_-(\mathbf{T}, F_0) < \infty$ coincides with the set of all generalized Caratheodory functions, i.e. with the union of all Krein-Langer classes $C_k(\mathcal{L}(\mathcal{H}))$, $k = 0, 1, \ldots$ (see [4]). This means that, roughly speaking, an open set Δ is of type π_+ with respect to G if and only if in a neighbourhood of Δ, G behaves similarly to a generalized Nevanlinna function. Analogously for F. In the following proposition this fact is more precisely expressed with the help of decompositions.

Proposition 2.8. *Let Δ be an open subset of $\Omega \cap \overline{\mathbf{R}}$. Then the following statements are equivalent.*

(i) *Δ is of type π_+ with respect to G.*
(i′) *$\psi(\Delta)$ is of type π_+ with respect to F.*

(ii) *For every open set Δ_0 which is the union of a finite number of pairwise disjoint connected open subsets of Δ such that $\overline{\Delta_0} \subset \Delta$, G can be written as a sum $G = G_0 + G_{(0)}$, where $G_0 \in N_k(\mathcal{L}(\mathcal{H}))$ for some k and $G_{(0)}$ is locally holomorphic on $\overline{\Delta}_0$.*

(ii′) *For every open set Γ_0 which is the union of a finite number of pairwise disjoint connected open subsets of $\psi(\Delta)$ such that $\overline{\Gamma}_0 \subset \psi(\Delta)$, F can be written as a sum $F = F_0 + F_{(0)}$, where $F_0 \in C_k(\mathcal{L}(\mathcal{H}))$ for some k and $F_{(0)}$ is locally holomorphic on $\overline{\Gamma}_0$.*

Proof. That (ii′) implies (i′) is proved as in the proof of Proposition 2.6, (iv′) $\Longrightarrow$ (iii′).

Assume that (i′) holds. Then we again construct a decomposition of F as in the proof of Proposition 2.1 and assume, in addition, that α in (2.10) is the square of a nonnegative function $\beta \in C_0^\infty(\psi(\Delta))$. Then by (2.6) and (2.5) we have $T_{F_0} \,.\, f = T_F \,.\, \beta^2 f$ for all $f \in C^\infty(\mathbf{T})$ and, by approximating functions from $C^\infty(\mathbf{T}, \mathcal{H})$ by $\mathcal{H}$-valued trigonometric polynomials,

$$T_{F_0}(u, v).\mathbf{1} = T_F(\beta u, \beta v).\mathbf{1}$$

for all $u, v \in C^\infty(\mathbf{T}, \mathcal{H})$. By condition (i′) the form $T_F(\beta\cdot, \beta\cdot).\mathbf{1}$ has a finite number of negative squares and F_0 is a generalized Caratheodory function, i.e. (ii′) is true. The rest of the proof is an immediate consequence of the above considerations. □

2.5. Locally definitizable operator functions

In the following definitions we define classes of operator functions which contain those considered in Proposition 2.8.

Definition 2.9. G is called *definitizable in* Ω if the following holds.

(α) For every finite union Δ_0 of open connected subsets of $\Omega \cap \overline{\mathbf{R}}$ with $\overline{\Delta_0} \subset \Omega \cap \overline{\mathbf{R}}$ there exists a positive integer m such that the order of growth of G near Δ_0 is $\leq m$.

(β) Every point $\lambda \in \Omega \cap \overline{\mathbf{R}}$ has an open connected neighbourhood I_λ in $\overline{\mathbf{R}}$ such that both components of $I_\lambda \setminus \{\lambda\}$ are of definite type with respect to G.

Definition 2.9′**.** F is called *definitizable in* $\psi(\Omega)$ if the following holds.

(α') For every finite union Γ_0 of open arcs of $\psi(\Omega) \cap \mathbf{T}$ with $\overline{\Gamma}_0 \subset \psi(\Omega) \cap \mathbf{T}$ there exists a positive integer m such that the order of growth of F near Γ_0 is $\leq m$.

(β') Every point $z \in \psi(\Omega) \cap \mathbf{T}$ has an open connected neighbourhood I_z in $\mathbf{T}$ such that both components of $I_z \setminus \{z\}$ are of definite type with respect to F.

Similarly to Proposition 2.6 we obtain the following proposition. For characterizations of functions definitizable in $\overline{\mathbf{C}}$ (which occur in the assertions (2) and (2′) below) with the help of definitizing rational functions see [4], [5].

Proposition 2.10. *The following statements are equivalent.*

(1) *G is definitizable in Ω.*

(1′) *F is definitizable in $\psi(\Omega)$.*

(2) *For every open set* $\Delta \subset \Omega \cap \overline{\mathbf{R}}$ *which is the union of a finite number of pairwise disjoint connected open subsets of* $\Omega \cap \overline{\mathbf{R}}$ *such that* $\overline{\Delta} \subset \Omega \cap \overline{\mathbf{R}}$, G *can be written as a sum* $G = G_0 + G_{(0)}$, *where* G_0 *is an* $\overline{\mathbf{R}}$*-symmetric* $\mathcal{L}(\mathcal{H})$*-valued function definitizable in* $\overline{\mathbf{C}}$, *and* $G_{(0)}$ *is locally holomorphic on* $\overline{\Delta}$.

(2′) *For every open set* $\Gamma \subset \psi(\Omega) \cap \mathbf{T}$ *which is the union of a finite number of pairwise disjoint connected open subsets of* $\psi(\Omega) \cap \mathbf{T}$ *such that* $\overline{\Gamma} \subset \psi(\Omega) \cap \mathbf{T}$, F *can be written as a sum* $F = F_0 + F_{(0)}$, *where* F_0 *is an* $\mathbf{T}$*-symmetric* $\mathcal{L}(\mathcal{H})$*-valued function definitizable in* $\overline{\mathbf{C}}$, *and* $F_{(0)}$ *is locally holomorphic on* $\overline{\Gamma}$.

For a function G definitizable in Ω we can even find an essentially unique decomposition of G similar to that in Proposition 2.10, (2), if we make some further requirements. Exactly the same is true for F. We shall formulate and prove it only for G.

Proposition 2.11. *Let* G *be definitizable in* Ω *and let* Δ *be as in Proposition* 2.10, (2), *and assume, additionally, that the endpoints of the connected components of* Δ *are finite and do not belong to* $\mathrm{K}(G, \Omega)$. *Moreover, let* Ω' *be a domain in* $\overline{\mathbf{C}}$ *with the same properties as* Ω *such that* $\overline{\Omega'} \subset \Omega$. *Then* G *can be written as a sum*

$$G = G_1 + G_2 + G_3, \tag{2.15}$$

where

(a) G_1 *is an* $\overline{\mathbf{R}}$*-symmetric* $\mathcal{L}(\mathcal{H})$*-valued function definitizable in* $\overline{\mathbf{C}}$ *and locally holomorphic in* $\overline{\mathbf{C}} \setminus \overline{\Delta}$. *If* t_0 *is an endpoint of a connected component of* Δ, *then* $t_0 \notin \mathrm{K}(G_1, \Omega)$ *and for every* $x \in \mathcal{H}$ *the angular limit*

$$\widehat{\lim}_{\lambda \to t_0} \, (\lambda - t_0)[G_1(\lambda)x, x]$$

is zero.

(b) G_2 *is a meromorphic* $\overline{\mathbf{R}}$*-symmetric* $\mathcal{L}(\mathcal{H})$*-valued function in* $\overline{\mathbf{C}}$ *with all poles contained in* $\Omega' \setminus \overline{\mathbf{R}}$.

(c) G_3 *is an* $\overline{\mathbf{R}}$*-symmetric* $\mathcal{L}(\mathcal{H})$*-valued function which is locally holomorphic on* $(\Omega' \setminus \overline{\mathbf{R}}) \cup \Delta$.

For fixed Δ *and* Ω' *as above, the terms of the decomposition* (2.15) *are uniquely determined up to addition of bounded selfadjoint operators.*

Proof. Let Δ_0 be an open subset of $\Omega \cap \overline{\mathbf{R}}$ with the same properties as Δ in Proposition 2.10 and assume that $\overline{\Delta} \subset \Delta_0$. We consider a decomposition $G = G_0 + G_{(0)}$ as in Proposition 2.10, (2), but with Δ replaced by Δ_0. Then the endpoints of the connected components of Δ do not belong to $\mathrm{K}(G_0, \overline{\mathbf{C}})$. Let $\lambda_0 \in \Omega' \cap \mathbf{C}^+$ be a point of holomorphy of G_0 and let A_0 be a minimal representing definitizable selfadjoint relation in some Krein space $\mathcal{K}$ for G_0:

$$G_0(\lambda) = S + \Gamma^+\{\lambda - \mathrm{Re}\,\lambda_0 + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A_0 - \lambda)^{-1}\}\Gamma, \quad \lambda \in \rho(A_0).$$

Here S is a bounded selfadjoint operator in $\mathcal{H}$ and $\Gamma \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. The endpoints of the components of Δ are no critical points of A_0 (see [5]). Then the spectral function $E(\cdot, A_0)$ of A_0 is defined on Δ and

$$G_1(\lambda) = \Gamma^+\{\lambda - \operatorname{Re}\,\lambda_0 + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0(A_0 - \lambda)^{-1}\}E(\Delta, A_0)\Gamma, \quad \lambda \notin \sigma(A_0) \cap \overline{\Delta},$$

is a definitizable $\mathcal{L}(\mathcal{H})$-valued function locally holomorphic in $\overline{\mathbf{C}} \setminus \overline{\Delta}$, and $G - G_1$ is locally holomorphic on Δ.

Let t_0 be an endpoint of a component of Δ. Then t_0 is no eigenvalue of $A_0 \cap (E(\Delta, A_0)\mathcal{K})^2$ and, therefore,

$$\begin{aligned}&\widehat{\lim}_{\lambda \to t_0}\ (\lambda - t_0)[\Gamma^+(A_0 - \lambda)^{-1}E(\Delta, A_0)\Gamma x, x]\\ &\qquad = \widehat{\lim}_{\lambda \to t_0}\ (\lambda - t_0)[(A_0 - \lambda)^{-1}E(\Delta, A_0)\Gamma x, \Gamma x] = 0,\end{aligned}$$

that is, G_1 fulfils condition (a).

Let $\mathcal{C}^+$ be a smooth simple closed curve in $\overline{\Omega'} \cap \mathbf{C}^+$ oriented in such a way that its interior domain is bounded. Assume that G is holomorphic on $\mathcal{C}^+$ and the set of all poles of G in the interior of $\mathcal{C}^+$ coincides with the set of all poles of G in $(\Omega' \setminus \overline{\mathbf{R}}) \cap \mathbf{C}^+$. By $\mathcal{C}^-$ we denote the curve $(\mathcal{C}^+)^*$ with the orientation opposite to that induced by $\mathcal{C}^+$. Let $\mathcal{C} := \mathcal{C}^+ \cup \mathcal{C}^-$.

We define

$$G_2(\lambda) := G(\lambda) - G_1(\lambda) - (2\pi i)^{-1}\int_{\mathcal{C}} (G(\mu) - G_1(\mu))(\mu - \lambda)^{-1}d\mu$$

$$G_3(\lambda) := G(\lambda) - G_1(\lambda) - G_2(\lambda) = (2\pi i)^{-1}\int_{\mathcal{C}} (G(\mu) - G_1(\mu))(\mu - \lambda)^{-1}d\mu.$$

It is easy to see that the functions G_2 and G_3 satisfy the conditions (b) and (c) of Proposition 2.11. The fact that G_2 is uniquely determined up to a bounded selfadjoint operator follows from Liouville's Theorem. Evidently, the difference $\tilde{G}_1$ of any two functions satisfying the conditions on G_1 is holomorphic in the complement of the set of the endpoints of the components of Δ. Since these endpoints are no critical points of the functions, the points of nonholomorphy of $\tilde{G}_1$ are poles of first order. Then it follows from the last condition in (a) that $\tilde{G}_1$ is a constant, and Proposition 2.11 is proved. □

3. Operator and relation representations of locally definitizable operator functions

3.1. Locally definitizable operator functions defined by locally definitizable relations

Let again Ω be a domain in $\overline{\mathbf{C}}$ with the properties mentioned in the introduction, $\lambda_0 \in \Omega \cap \mathbf{C}^+$, and let besides the Krein space $\mathcal{H}$, $\mathcal{K}$ be a further Krein space. We recall the definition of local definitizability for selfadjoint relations and unitary operators in $\mathcal{K}$ from [6, Definition 4.4]. For equivalent descriptions of locally definitizable relations see [6, Theorem 4.8].

Definition 3.1. The selfadjoint relation A with $\lambda_0 \in \rho(A)$ (the unitary operator U) is called *definitizable over* Ω (resp. *definitizable over* $\psi(\Omega)$) if $\sigma(A) \cap (\Omega \setminus \overline{\mathbf{R}})$ (resp. $\sigma(U) \cap (\psi(\Omega) \setminus \mathbf{T})$) consists of isolated points which are poles of the resolvent and the function

$$\lambda \longmapsto \lambda - \mathrm{Re}\,\lambda_0 + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A - \lambda)^{-1}$$
$$(\text{resp. } z \longmapsto (U + z)(U - z)^{-1})$$

is definitizable in Ω (resp. definitizable in $\psi(\Omega)$).

If A is a selfadjoint relation in $\mathcal{K}$ with $\lambda_0 \in \rho(A)$ and U is the unitary operator defined by

$$U := \psi(A) = -1 + (\lambda_0 - \bar{\lambda}_0)(A - \bar{\lambda}_0)^{-1}, \tag{3.1}$$

then

$$-\, i(\mathrm{Im}\lambda_0)^{-1}\{\lambda - \mathrm{Re}\,\lambda_0 + (\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)(A - \lambda)^{-1}\} = (U + \psi(\lambda))(U - \psi(\lambda))^{-1}. \tag{3.2}$$

Therefore, A is definitizable over Ω if and only if U is definitizable over $\psi(\Omega)$.

Now let A be definitizable over Ω and $\lambda_0 \in \rho(A) \cap \Omega \cap \mathbf{C}^+$. We denote the local spectral function of A ([6]) which is defined on a collection of subsets of $\Omega \cap \overline{\mathbf{R}}$ by $E(\cdot, A)$. If ω is a subset of $\Omega \setminus \overline{\mathbf{R}}$ such that $\omega \cap \sigma(A)$ is closed and open in $\sigma(A)$, the same notation will be used to denote the Riesz-Dunford projection corresponding to $\omega \cap \sigma(A)$: $E(\omega, A)$. Let S be a bounded selfadjoint operator in $\mathcal{H}$ and let $\Gamma \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. We consider the $\mathcal{L}(\mathcal{H})$-valued function G defined by

$$G(\lambda) = S + \Gamma^+\{\lambda - \mathrm{Re}\,\lambda_0 + (\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)(A - \lambda)^{-1}\}\Gamma, \quad \lambda \in \rho(A) \cap \Omega. \tag{3.3}$$

Then, by Definition 2.9, also G is definitizable in Ω. If an operator function G can be written as in (3.3), A is called a *representing relation for* G. In this case, evidently,

$$S = \frac{1}{2}(G(\lambda_0) + G(\lambda_0)^+) =: \mathrm{Re}^+\, G(\lambda_0).$$

The representation (3.3) is called *minimal* if

$$\mathcal{K} = \mathrm{clsp}\,\{(1 + (\lambda - \lambda_0)(A - \lambda)^{-1})\Gamma y :\ \lambda \in \rho(A) \cap \Omega,\ y \in \mathcal{H}\}.$$

Similarly, if U is a unitary operator in $\mathcal{K}$ definitizable over $\psi(\Omega)$, S_0 is a bounded selfadjoint operator in $\mathcal{H}$ and $\Gamma_0 \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, then the function F defined by

$$F(z) = -iS_0 + \Gamma_0^+(U + z)(U - z)^{-1}\Gamma_0 \tag{3.4}$$

is definitizable in $\psi(\Omega)$. Observe that

$$-S_0 = (2i)^{-1}(F(0) - F(0)^+) =: \mathrm{Im}^+\, F(0).$$

If a relation of the form (3.4) holds, U is called a *representing operator for* F. The representation (3.4) is called *minimal* if

$$\mathcal{K} = \mathrm{clsp}\,\{U^m\Gamma y :\ m = 0, \pm 1, \pm 2, \ldots,\ y \in \mathcal{H}\}.$$

If

$$U = \psi(A), \quad S_0 = S, \quad \Gamma_0 = (\operatorname{Im} \lambda_0)^{\frac{1}{2}}\Gamma, \tag{3.5}$$

then, in view of (3.2), the functions G and F are connected by

$$-iG(\lambda) = F(\psi(\lambda)), \quad \lambda \in \Omega.$$

In the following definition we introduce a local version of minimality.

Definition 3.2. Let A be a selfadjoint relation definitizable over Ω in a Krein space $\mathcal{K}$ with $\lambda_0 \in \rho(A)$, $\Gamma \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and S a bounded selfadjoint operator in $\mathcal{H}$. Let G be defined by (3.3) and let U and F be as in (3.1) and (3.4).

Then (3.3) is called an Ω-*minimal representation of* G if the following holds: If Ω' is a domain with the same properties as Ω and $\overline{\Omega'} \subset \Omega$, $\lambda_0 \in \Omega'$, if Δ is a finite union of connected open subsets of $\Omega \cap \overline{\mathbf{R}}$ such that the endpoints of the components of Δ belong to Ω and possess open neighbourhoods of definite type with respect to A, and if we set

$$\tilde{E} := E(\Delta, A) + E(\overline{\Omega'} \setminus \overline{\mathbf{R}}, A), \tag{3.6}$$

then

$$\tilde{E}\mathcal{K} = \operatorname{clsp}\{(1 + (\lambda - \lambda_0)(A - \lambda)^{-1})\tilde{E}\Gamma y : \ \lambda \in \rho(A) \cap \Omega', \ y \in \mathcal{H}\}.$$

Similarly, (3.4) is called an $\psi(\Omega)$-*minimal representation of* F if for every projection $\tilde{E}$ as above (note that $\tilde{E}$ coincides with $E(\psi(\Delta), U) + E(\psi(\overline{\Omega'}) \setminus \mathbf{T}, U))$ we have

$$\tilde{E}\mathcal{K} = \operatorname{clsp}\{U^m \tilde{E}\Gamma y : \ m = 0, \pm 1, \pm 2, \ldots, \ y \in \mathcal{H}\}.$$

Evidently, if (3.3) is minimal, it is also Ω-minimal. If (3.3) is Ω-minimal, then it is Ω_0-minimal for every domain Ω_0 with the same properties as Ω and $\Omega_0 \subset \Omega$. The following lemma is an easy consequence of Definition 3.2.

Lemma 3.3. *Let G and F be as in Definition* 3.2. *Then the following statements are equivalent.*

(i) *The representation* (3.3) *is Ω-minimal.*

(ii) *The representation* (3.4) *is $\psi(\Omega)$-minimal.*

(iii) *For every Ω' and Δ as in Definition* 3.2, *with $\tilde{E}$ as in* (3.6) *and $\tilde{A} := A|\tilde{E}\mathcal{K}$, the function*

$$\lambda \longmapsto (\tilde{E}\Gamma)^+\{\lambda - \operatorname{Re} \lambda_0 + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(\tilde{A} - \lambda)^{-1}\}(\tilde{E}\Gamma) \tag{3.7}$$

is minimally represented by (3.7) *with $\tilde{A}$ as representing relation.*

(iv) *For every Ω' and Δ as in Definition* 3.2, (3.6) *and $\tilde{U} := U|\tilde{E}\mathcal{K}$, the function*

$$z \longmapsto (\tilde{E}\Gamma_0)^+(\tilde{U} + z)(\tilde{U} - z)^{-1}(\tilde{E}\Gamma_0) \tag{3.8}$$

is minimally represented by (3.8) *with $\tilde{U}$ as representing operator.*

Proof. Evidently, (i) is equivalent to (iii) and (ii) is equivalent to (iv). In order to show that (i) implies (ii) let $\lambda \in \rho(A) \cap \Omega' \cap \mathbf{C}^+$ and connect the point λ by a smooth curve in $\lambda \in \rho(A) \cap \Omega' \cap \mathbf{C}^+$ with the point λ_0. Making use of Taylor expansions of the resolvent of A at a finite number of points of this curve we see that every element of the form $(A-\lambda)^{-1}\tilde{E}\Gamma y$, $y \in \mathcal{H}$ can be approximated by linear combinations of elements of the form $(A-\lambda_0)^{-j}\tilde{E}\Gamma y_j$, $y_j \in \mathcal{H}$, $j = 0, 1, \ldots$. Since by (3.1)

$$(A-\lambda_0)^{-j} = (\overline{\lambda}_0 - \lambda_0)^{-j}(1+U^{-1})^j, \quad j = 0, 1, \ldots,$$

$(A-\lambda)^{-1}\tilde{E}\Gamma y$ can be approximated by linear combinations of elements of the form

$$U^{-k}\tilde{E}\Gamma u_k, \quad u_k \in \mathcal{H}, \quad k = 0, 1, \ldots.$$

Analogously for $\lambda \in \rho(A) \cap \Omega' \cap \mathbf{C}^-$. Therefore, (i) implies (ii).

On the other hand, every element of the form

$$U^{-m}\tilde{E}\Gamma u = (-1 + (\overline{\lambda}_0 - \lambda_0)(A-\lambda_0)^{-1})^m \tilde{E}\Gamma u, \quad m \in \mathbf{N}, \quad u \in \mathcal{H},$$

can be approximated by linear combinations of elements of the form $(A-\lambda_j)^{-1}\tilde{E}\Gamma y_j$ or $\tilde{E}\Gamma y_j$, $\lambda_j \in \rho(A) \cap \Omega' \cap \mathbf{C}^+$, $y_j \in \mathcal{H}$. Analogously for m replaced by $-m$. This shows that (i) is equivalent to (ii). □

In the following proposition the local "sign multiplicities" of a function and a representing relation are compared.

Proposition 3.4. *If A, U, G and F are as above in this section, then the following holds.*

(1) *Let Δ_0 be the union of a finite number of pairwise disjoint connected open subsets of $\Omega \cap \overline{\mathbf{R}}$ such that $\overline{\Delta}_0 \subset \Omega \cap \overline{\mathbf{R}}$ and $E(\Delta_0, A)$ and $E(\psi(\Delta_0), U)$ are defined. Then*

$$\begin{aligned}\kappa_\pm(\Delta_0, G) &= \kappa_\pm(\psi(\Delta_0), F) \\ &\le \kappa_\pm((E(\Delta_0, A)\mathcal{K}, [\cdot,\cdot])) = \kappa_\pm((E(\psi(\Delta_0), U)\mathcal{K}, [\cdot,\cdot])).\end{aligned} \tag{3.9}$$

If, in addition, the representations (3.3) and (3.4) are Ω-minimal and $\psi(\Omega)$-minimal, respectively, we have equality in (3.9).

(2) *Let $\mu \in \Omega \setminus \overline{\mathbf{R}}$ be a pole of G of multiplicity l, or equivalently, let $\psi(\mu) \in \psi(\Omega) \setminus \mathbf{T}$ be a pole of F of multiplicity l. Then*

$$l \le \dim\ E(\{\mu\}, A)\mathcal{K} = \dim\ E(\{\psi(\mu)\}, U)\mathcal{K}, \tag{3.10}$$

where $E(\{\mu\}, A)$ and $E(\{\psi(\mu)\}, U)$ denote the Riesz-Dunford projections corresponding to A and $\{\mu\}$, and to U and $\psi(\mu)$, respectively. Under the condition mentioned in (1) we have equality in (3.10).

Proof. 1. By (2.14) it is sufficient to prove (3.9) for F and U. If F_0 and $F_{(0)}$ are the functions defined by

$$\begin{aligned}F_0(z) &= \Gamma_0^+(U+z)(U-z)^{-1}E(\psi(\Delta_0), U)\Gamma_0, \\ F_{(0)}(z) &= -iS_0 + \Gamma_0^+(U+z)(U-z)^{-1}(1 - E(\psi(\Delta_0), U))\Gamma_0,\end{aligned}$$

which are definitizable over $\psi(\Omega)$, then we have $F = F_0 + F_{(0)}$. By Lemma 2.4 the forms $T_F(\cdot,\cdot)$ and $T_{F_0}(\cdot,\cdot)$ coincide on $\psi(\Delta_0)$. Therefore,

$$\kappa_\pm(\psi(\Delta_0), F) = \kappa_\pm(\psi(\Delta_0), F_0). \tag{3.11}$$

Since F_0 is a definitizable function [5, Theorem 1.12,(iii)] can be applied. We find

$$\kappa_\pm(\psi(\Delta_0), F_0) = \kappa_\pm((E(\psi(\Delta_0), U)\mathcal{K}, [\cdot,\cdot])), \tag{3.12}$$

and equality holds if the representation (3.4) is Ω-minimal (see Lemma 3.3, (ii) $\Leftrightarrow$ (iv)). The relations (3.11) and (3.12) imply assertion (1).

2. To prove (2) it is again sufficient to verify (2) for F and U. If $E_1 := E(\{\psi(\mu)\} \cup \{\overline{\psi(\mu)}^{-1}\}, U)$ and

$$F_1(z) = \Gamma_0^+(U+z)(U-z)^{-1}E_1\Gamma_0, \tag{3.13}$$
$$F_{(1)}(z) = -iS_0 + \Gamma_0^+(U+z)(U-z)^{-1}(1-E_1)\Gamma_0,$$

then $F = F_1 + F_{(1)}$, F_1 is a definitizable function and $\psi(\mu)$ is a pole of multiplicity l of F_1. Then [5, Theorem 1.12,(iv)] implies

$$l \leq \dim\ E(\{\psi(\mu)\}, U)\mathcal{K}.$$

If the representation (3.4) is Ω-minimal, then (3.13) is a minimal representation of F_1 and, by the result mentioned above we have equality. □

In the rest of Section 3.1 we consider two Ω-minimal representing relations A_1 and A_2 of the same operator function G. By Proposition 3.4 the local "sign properties inside Ω" of A_1 and A_2 coincide. In Theorem 3.6 below we will show that the restrictions of A_1 and A_2 to spectral subspaces which correspond to certain subsets of $\Omega \cap \overline{\mathbf{R}}$ are even unitarily equivalent. We need the following lemma.

Lemma 3.5. *Let $(\mathcal{K}_j, [\cdot,\cdot])$, $j = 1, 2$, be Krein spaces and U_j, $j = 1, 2$, unitary operators in $\mathcal{K}_j$ definitizable over $\psi(\Omega)$, $\Gamma_{0,j} \in \mathcal{L}(\mathcal{H}, \mathcal{K}_j)$ and $S_{0,j}$ bounded selfadjoint operators in $\mathcal{H}$, $j = 1, 2$.*

We denote by Ξ the linear space of all functions χ defined on the union of $\psi(\Omega) \cap \mathbf{T}$ and a neighbourhood $\mathcal{U}$ (depending on χ) of $(\overline{\mathbf{C}} \setminus \psi(\Omega)) \cup \sigma(U_1) \cup \sigma(U_2)$ which are sums $\chi = \chi_{\mathbf{T}} + \chi_{(\mathbf{T})}$ of a function $\chi_{\mathbf{T}} \in C_0^\infty(\psi(\Omega) \cap \mathbf{T})$ and a function $\chi_{(\mathbf{T})}$ locally holomorphic on $(\overline{\mathbf{C}} \setminus \psi(\Omega)) \cup \sigma(U_1) \cup \sigma(U_2) \cup \mathbf{T}$ which is zero outside of some compact subset of $\psi(\Omega) \setminus \mathbf{T}$.

Assume that the difference of the functions

$$\begin{gathered} F_j(z) := -iS_{0,j} + \Gamma_{0,j}^+(U_1+z)(U_j-z)^{-1}\Gamma_{0,j}, \\ j = 1, 2, \quad z \in \rho(U_1) \cap \rho(U_2) \cap (\psi(\Omega) \setminus \mathbf{T}), \end{gathered} \tag{3.14}$$

can be analytically continued to the whole domain $\psi(\Omega)$.

Then the linear relation

$$\begin{aligned} V := &\left\{ \begin{pmatrix} \sum_{k=1}^n \chi_k(U_1)\Gamma_{0,1}x_k \\ \sum_{k=1}^n \chi_k(U_2)\Gamma_{0,2}x_k \end{pmatrix} : \ \chi_k \in \Xi,\ x_k \in \mathcal{H},\ k = 1, 2, ..., n \right\} \\ &\subset \mathcal{K}_1 \times \mathcal{K}_2 \end{aligned} \tag{3.15}$$

is isometric, i.e. $\binom{u_1}{u_2}, \binom{u_1'}{u_2'} \in V$ implies

$$[u_1, u_1']_1 = [u_2, u_2']_2. \tag{3.16}$$

Moreover, V intertwines U_1 and U_2, i.e. $\binom{u_1}{u_2} \in V$ implies $\binom{U_1u_1}{U_2u_2} \in V$.

Proof. If we denote by $\mathcal{R}_{0,\infty}$ the set of all functions $z \longmapsto \sum_k c_k z^k$, k integer, where the sums are finite, then, by [5, Section 1.3], for $g \in \mathcal{R}_{0,\infty}$ we have

$$T_{F_1}.g = 4\pi\Gamma_{0,1}^+ g(U_1)\Gamma_{0,1}, \quad T_{F_2}.g = 4\pi\Gamma_{0,2}^+ g(U_2)\Gamma_{0,2}.$$

By continuity properties of T_{F_1}, T_{F_2} and of the functional calculi of U_1 and U_2 these relations remain true for g replaced by an arbitrary $\chi \in \Xi$.

By the definition of V and since Ξ is an algebra contained in the domains of the functional calculi for U_1 and U_2, the left hand side of (3.16) is a finite sum of the form

$$\sum_{i,j} ((T_{F_1}.\chi_{i,j})x_i, y_j), \quad \chi_{i,j} \in \Xi, \quad x_i, y_j \in \mathcal{H}. \tag{3.17}$$

Then the right hand side of (3.16) coincides with

$$\sum_{i,j} ((T_{F_2}.\chi_{i,j})x_i, y_j). \tag{3.18}$$

Since the difference of F_1 and F_2 can be analytically continued to $\psi(\Omega)$, the expressions (3.17) and (3.18) coincide, which shows that V is isometric. That V intertwines U_1 and U_2 follows from the definition of V and the fact that for $\chi \in \Xi$ also the function $z \longmapsto z\chi(z)$ belongs to Ξ. Lemma 3.5 is proved. □

Theorem 3.6. *Let $(\mathcal{K}_j, [\cdot,\cdot]_j)$, $j = 1, 2$, be Krein spaces and A_j, $j = 1, 2$, selfadjoint relations in $\mathcal{K}_j$ definitizable over Ω, let $\lambda_0 \in (\Omega \cap \mathbf{C}^+) \cap \rho(A_1) \cap \rho(A_2)$, $\Gamma_j \in \mathcal{L}(\mathcal{H}, \mathcal{K}_j)$ and S_j bounded selfadjoint operators in $\mathcal{H}$, $j = 1, 2$. Assume that the difference of the functions G_j defined by*

$$\begin{aligned} G_j(\lambda) :=& S_j + \Gamma_j^+\{\lambda - \mathrm{Re}\lambda_0 + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A_j - \lambda)^{-1}\}\Gamma_j, \\ & j = 1, 2, \quad \lambda \in \rho(A_1) \cap \rho(A_2) \cap (\Omega \setminus \mathbf{R}), \end{aligned} \tag{3.19}$$

can be analytically continued to the whole domain Ω.

If $U_j := \psi(A_j)$, $F_j := -iG_j \circ \phi$, $S_{0,j} := S_j$, $\Gamma_{0,j} := (\mathrm{Im}\ \lambda_0)^{\frac{1}{2}}\Gamma_j$, $j = 1, 2$, then the above assumptions are equivalent to the assumptions of Lemma 3.5.

Assume further that the representations (3.19) *of G_1 and G_2 are Ω-minimal or, equivalently, that the representations* (3.14) *of F_1 and F_2 are $\psi(\Omega)$-minimal Then the following holds.*

(i) *An open set $\Delta \subset \Omega \cap \overline{\mathbf{R}}$ ($\Gamma \subset \psi(\Omega) \cap \mathbf{T}$) is of positive type with respect to A_1 (resp. U_1), that is, the spectral function $E(\cdot, A_1)$ (resp. $E(\cdot, U_1)$) is defined on all connected subsets of Δ (resp. Γ) with endpoints in Δ (resp. Γ) and its values are nonnegative projections in $\mathcal{K}_1$, if and only if it is of positive type with respect to A_2 (resp. U_2). Analogously for sets of negative type, that is, nonnegativity of the spectral projections replaced by nonpositivity.*

(ii) *Let Δ' be an open connected subset of $\Omega \cap \overline{\mathbf{R}}$ with $\overline{\Delta'} \subset \Omega \cap \overline{\mathbf{R}}$ such that the endpoints of Δ' are contained in intervals of positive or negative type. Then there exists a densely defined closed isometric operator V' from $E_1'\mathcal{K}_1$ into $E_2'\mathcal{K}_2$, where $E_j' := E(\Delta', A_j) = E(\psi(\Delta'), U_j)$, $j = 1, 2$, with dense range which intertwines the resolvents of $A_j' := A_j \cap (E_j'\mathcal{K}_j)^2$ as well as the operators*

$U'_j := U_j | E'_j \mathcal{K}$, $i = 1, 2$, *i.e. for* $\lambda \notin \Delta'$ *we have* $V'(A'_1 - \lambda)^{-1} = (A'_2 - \lambda)^{-1} V'$, $V' U'_1 = U'_2 V'$. *In particular, we have*

$$\kappa_\pm((E'_1 \mathcal{K}_1, [\cdot,\cdot]_1)) = \kappa_\pm((E'_2 \mathcal{K}_2, [\cdot,\cdot]_2)).$$

(iii) *If, in addition to the assumptions of* (ii), $\kappa_+((E'_1\mathcal{K}_1, [\cdot,\cdot]_1)) < \infty$, *then* A'_1 *and* A'_2 *(*U'_1 *and* U'_2*) are isometrically equivalent, that is, there exists an operator* V' *as in* (ii) *which is even an isometric isomorphism of* $(E'_1\mathcal{K}_1, [\cdot,\cdot]_1)$ *onto* $(E'_2\mathcal{K}_2, [\cdot,\cdot]_2)$.

(iv) *If* $\mu \in \Omega \setminus \overline{\mathbf{R}}$ *is a pole of* G_1 *and* G_2 *or, equivalently,* $\psi(\mu) \in \psi(\Omega) \setminus \mathbf{T}$ *is a pole of* F_1 *and* F_2*, then there exists an injective densely defined closed operator* V_μ *from* $\mathcal{K}_{1,\mu} := E(\{\mu\}, A_1)\mathcal{K}_1 = E(\{\psi(\mu)\}, U_1)\mathcal{K}_1$ *into* $\mathcal{K}_{2,\mu} := E(\{\mu\}, A_2)\mathcal{K}_2 = E(\{\psi(\mu)\}, U_2)\mathcal{K}_2$ *with dense range such that* $A_1\mathcal{D}(V_\mu) \subset \mathcal{D}(V_\mu)$, $U_1\mathcal{D}(V_\mu) \subset \mathcal{D}(V_\mu)$ *and* $V_\mu A_1 x = A_2 V_\mu x$, $V_\mu U_1 x = U_2 V_\mu x$ *for all* $x \in \mathcal{D}(V_\mu)$.

Proof. It is sufficient to prove Theorem 3.6 for U_1 and U_2. Assertion (i) is an immediate consequence of Proposition 3.4, (1).

Let Δ' be as in assertion (ii). By the minimality assumptions the linear set

$$\mathrm{sp}\,\{h(U_j)\Gamma_{0,j}x : \ h \in C_0^\infty(\psi(\Delta')), \ x \in \mathcal{H}\}$$

is dense in $E(\psi(\Delta'), U_j)\mathcal{K}_j$, $j = 1, 2$. If V is the linear relation introduced in Lemma 3.5, the relation

$$V'_0 := V \cap (E(\psi(\Delta'), U_1)\mathcal{K}_1 \times E(\psi(\Delta'), U_2)\mathcal{K}_2)$$

is densely defined in $E(\psi(\Delta'), U_1)\mathcal{K}_1$ and has dense range in $E(\psi(\Delta'), U_2)\mathcal{K}_2$. Since V'_0 is isometric (see Lemma 3.5) it is even a closable operator. Let V' be the closure of V'_0. Then V' is also isometric. The intertwining properties of V imply the intertwining properties of V' mentioned in (ii). Assertion (iii) is a consequence of the fact that an isometric operator from a Pontryagin space into a Pontryagin space with dense domain and dense range is an isometric isomorphism.

If μ is as in assertion (iv), then by the minimality assumptions the relation

$$V_{\mu,\bar\mu;0} := V \cap (E(\{\psi(\mu), \psi(\bar\mu)\}, U_1)\mathcal{K}_1 \times E(\{\psi(\mu), \psi(\bar\mu)\}, U_2)\mathcal{K}_2)$$

is isometric, densely defined in $E(\{\psi(\mu), \psi(\bar\mu)\}, U_1)\mathcal{K}_1$ and has dense range in $E(\{\psi(\mu), \psi(\bar\mu)\}, U_2)\mathcal{K}_2$. Therefore, $V_{\mu,\bar\mu;0}$ is a closable operator. Let $V_{\mu,\bar\mu}$ be its closure. From the definition of V it follows that

$$\mathcal{D}(V_{\mu,\bar\mu;0}) = \mathcal{D}(V_{\mu,\bar\mu;0}) \cap E(\{\psi(\mu)\}, U_1)\mathcal{K}_1 + \mathcal{D}(V_{\mu,\bar\mu;0}) \cap E(\{\psi(\bar\mu)\}, U_1)\mathcal{K}_1, \quad (3.20)$$

and the intersections on the right hand side of (3.20) are dense in $E(\{\psi(\mu)\}, U_1)\mathcal{K}_1$ and $E(\{\psi(\bar\mu)\}, U_1)\mathcal{K}_1$, respectively. Analogously for the range of $V_{\mu,\bar\mu;0}$ with U_1, $\mathcal{K}_1$ replaced by U_2, $\mathcal{K}_2$. Moreover, $V_{\mu,\bar\mu;0}$ maps the first intersection on the right hand side of (3.20) into $E(\{\psi(\mu)\}, U_2)\mathcal{K}_2$ and the second into $E(\{\psi(\bar\mu)\}, U_2)\mathcal{K}_2$. Then the closure $V_{\mu,\bar\mu}$ has analogous properties, and assertion (iv) is true with V_μ being the restiction of $V_{\mu,\bar\mu}$ to $E(\{\psi(\mu)\}, U_1)\mathcal{K}_1$. Theorem 3.6 is proved. $\square$

3.2. Existence of a locally definitizable representing relation with a local minimality property

In this section we shall construct representing relations for given locally definitizable operator functions. In the following theorem, which is a variant of a result of T. Ya. Azizov ([1]), we consider operator functions holomorphic in Ω and $\psi(\Omega)$. We show that for a given neighbourhood of $\overline{\mathbf{C}}\setminus\Omega$ or $\overline{\mathbf{C}}\setminus\psi(\Omega)$ there exist representing operators or relations the spectrum of which is contained in that neighbourhood. The extended spectrum of a relation T in a Krein space $\mathcal{K}$ will be denoted by $\widetilde{\sigma}(T)$, i.e., $\widetilde{\sigma}(T)=\sigma(T)$ if $T\in\mathcal{L}(\mathcal{K})$ and $\widetilde{\sigma}(T)=\sigma(T)\cup\{\infty\}$ if $T\notin\mathcal{L}(\mathcal{K})$. We set $\widetilde{\rho}(T)=\overline{\mathbf{C}}\setminus\widetilde{\sigma}(T)$.

Theorem 3.7. *Let $\mathcal{V}$ be an open neighbourhood of $\overline{\mathbf{C}}\setminus\Omega$ and let $\lambda_0\in\mathbf{C}^+\cap(\Omega\setminus\overline{\mathcal{V}})$. Let G be an $\mathcal{L}(\mathcal{H})$-valued function holomorphic in Ω such that $G=G^*$.*

Then there exist a Krein space $\mathcal{K}$, a selfadjoint relation A in $\mathcal{K}$ and $\Gamma\in\mathcal{L}(\mathcal{H},\mathcal{K})$ such that

$$\widetilde{\sigma}(A)\subset\mathcal{V}$$

and

$$G(\lambda)=\operatorname{Re}^+G(\lambda_0)+\Gamma^+\{\lambda-\operatorname{Re}\lambda_0+(\lambda-\lambda_0)(\lambda-\bar{\lambda}_0)(A-\lambda)^{-1}\}\Gamma,$$

$\lambda\in\Omega\setminus\overline{\mathcal{V}}$, or, equivalently, with $\psi(\lambda):=-(\lambda-\lambda_0)(\lambda-\overline{\lambda_0})^{-1}$, $F(\psi(\lambda)):=-iG(\lambda)$, $U:=\psi(A)$, $\Gamma_0:=(\operatorname{Im}\lambda_0)^{\frac12}\Gamma$,

$$\sigma(U)\subset\psi(\mathcal{V}) \tag{3.21}$$

and

$$F(z)=i\operatorname{Im}^+F(0)+\Gamma_0^+(U+z)(U-z)^{-1}\Gamma_0,\quad z\in\psi(\Omega)\setminus\psi(\overline{\mathcal{V}}). \tag{3.22}$$

Moreover, $\mathcal{K}$ can be chosen minimal, that is

$$\begin{aligned}\mathcal{K}&=\operatorname{closp}\{(1+(\lambda-\lambda_0)(A-\lambda)^{-1}\Gamma x:\ \lambda\in\Omega\setminus\overline{\mathcal{V}},x\in\mathcal{H}\}\\&=\operatorname{closp}\{U^m\Gamma_0x:\ m=0,\pm1,\pm2,\dots,x\in\mathcal{H}\}.\end{aligned} \tag{3.23}$$

Proof. It is sufficient to prove the assertions for F and U. We may and will assume that the set $\psi(\mathcal{V})$ is bounded and $\mathbf{T}$-symmetric. We set

$$d:=\inf\{|z-w|:\ z\in\overline{\mathbf{C}}\setminus\psi(\Omega),\ w\in\mathbf{C}\setminus\psi(\mathcal{V})\}.$$

Then with the help of a $\mathbf{T}$-symmetric covering of the bounded set $\overline{\mathbf{C}}\setminus\psi(\Omega)$ by a finite number of open disc neighbourhoods of points of $\overline{\mathbf{C}}\setminus\psi(\Omega)$ with radius $\le\frac12 d$ it is not difficult to find an open neighbourhood $\mathcal{W}$ of $\overline{\mathbf{C}}\setminus\psi(\Omega)$ with the following properties: (a) $\overline{\mathcal{W}}\subset\psi(\mathcal{V})$, (b) $\overline{\mathbf{C}}\setminus\overline{\mathcal{W}}$ is a piecewise analytic $\mathbf{T}$-symmetric domain of $\overline{\mathbf{C}}$, (c) $\mathbf{D}\setminus\overline{\mathcal{W}}$ is simply connected, (d) $\overline{\mathcal{W}}\cap\mathbf{T}$ consists of a finite number of pairwise disjoint closed arcs of $\mathbf{T}$. Observe that to find $\mathcal{W}$ with the property (c) the fact that $\mathbf{D}\cap\psi(\Omega)$ is simply connected has to be used. Then with the help of a conformal mapping of $\mathbf{D}\setminus\overline{\mathcal{W}}$ onto $\mathbf{D}$ and its $\mathbf{T}$-symmetric continuation it is easy to see that there exist bounded simply connected domains O_i, $i=1,\dots,n$, with analytic boundaries and the following properties: (a′) $O_i=\hat{O}_i$, $i=1,\dots,n$. (b′) The closures $\overline{O}_i$, $i=1,\dots,n$, are pairwise disjoint. (c′) $\overline{\mathbf{C}}\setminus\psi(\Omega)\subset O:=\bigcup_{i=1}^n O_i$.

(d′) $\overline{O} \subset \psi(\mathcal{V})$. Then F is an $\mathcal{L}(\mathcal{H})$-valued function which is locally holomorphic on $\overline{\mathbf{C}} \setminus O$ such that $F = -\hat{F}$.

For $u, v \in H(\overline{O}, \mathcal{H})$ we define the positive definite inner product

$$(u, v)_{H^2} := \int_{\partial O} (u(z), v(z))|dz|,$$

and we denote by $H^2(\overline{O}, \mathcal{H})$ the Hilbert space obtained by completion of $H(\overline{O}, \mathcal{H})$ with respect to $\|\cdot\|_{H^2}$, where $\|u\|_{H^2} := (u, u)_{H^2}^{\frac{1}{2}}$, $u \in H(\overline{O}, \mathcal{H})$. If we identify the space $H(\overline{O}, \mathcal{H})$ with the product $H(\overline{O}_1, \mathcal{H}) \times \cdots \times H(\overline{O}_n, \mathcal{H})$ and if Θ_i, $i = 1, \cdots, n$, is a conformal mapping of O_i on the unit disc $\mathbf{D}$, then the linear mapping

$$\mathbf{\Theta} : (H(\overline{\mathbf{D}}, \mathcal{H}))^n \ni (f_1, \cdots, f_n)^T \longmapsto (f_1 \circ \Theta_1, \cdots, f_n \circ \Theta_n)^T \in H(\overline{O}, \mathcal{H})$$

is bijective. It is easy to see that $\mathbf{\Theta}$ can be extended by continuity to an isomorphism $\widetilde{\mathbf{\Theta}}$ of the product $(H^2(\mathcal{H}))^n$ of the usual H^2-spaces $H^2(\mathcal{H})$ of $\mathcal{H}$-valued functions and $H^2(\overline{O}, \mathcal{H})$. Making use of the isomorphism $\widetilde{\mathbf{\Theta}}$ and well-known results on H^2-spaces (see e.g. [8, Section V, §1]) we see that $H^2(\overline{O}, \mathcal{H})$ can be regarded as a Hilbert subspace of the linear space of all locally holomorphic $\mathcal{H}$-valued functions in O such that for every compact subset K of O we have

$$\sup\{\|u(\lambda)\| : \lambda \in K, \; u \in H^2(\overline{O}, \mathcal{H}), \; \|u\|_{H^2} \leq 1\} < \infty. \tag{3.24}$$

Let $O_{0,i}$, $i = 1, \ldots, n$, be smooth domains such that $\overline{O}_{0,i} \subset O_i$ and F is still locally holomorphic on $\overline{\mathbf{C}} \setminus O_0$, $O_0 := \bigcup_{i=1}^n O_{0,i}$. Then we define, for $u, v \in H^2(\overline{O}, \mathcal{H})$,

$$[u, v]_0 := -\int_{\partial O_0} [F(z)u(z), v(\bar{z}^{-1})](iz)^{-1} dz.$$

By (3.24) $[\cdot, \cdot]_0$ is a continuous hermitian sesquilinear form on $H^2(\overline{O}, \mathcal{H})$. Let W be the Gram operator of $[\cdot, \cdot]_0$ in $H^2(\overline{O}, \mathcal{H})$ and let P_0 be the orthogonal projection in $H^2(\overline{O}, \mathcal{H})$ on the orthogonal complement (in $H^2(\overline{O}, \mathcal{H})$) of ker W. Let $\mathcal{K}$ be the completion of $P_0 H^2(\overline{O}, \mathcal{H})$ with respect to the restriction of the quadratic norm $\||W|^{\frac{1}{2}} \cdot \|_{H^2}$ to $P_0 H^2(\overline{O}, \mathcal{H})$. Evidently, the form $[\cdot, \cdot]_0$ can be extended by continuity to a form $[\cdot, \cdot]_{\mathcal{K}}$ in $\mathcal{K}$ and $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$ is a Krein space.

Let U' and V' be the operators of multiplication by z and z^{-1}, respectively, in the Hilbert space $H^2(\overline{O}, \mathcal{H})$. These operators are bounded and we have

$$[U' u_1, u_2]_0 = [u_1, V' u_2]_0, \qquad u_1, u_2 \in H^2(\overline{O}, \mathcal{H}).$$

Therefore, U' ker $W \subset$ ker W, V' ker $W \subset$ ker W, and, if we define bounded operators U_0, V_0 in $P_0 H^2(\overline{O}, \mathcal{H})$ by

$$U_0 := P_0 U' | P_0 H^2(\overline{O}, \mathcal{H}), \quad V_0 := P_0 V' | P_0 H^2(\overline{O}, \mathcal{H})$$

we find $U_0 V_0 = V_0 U_0 = 1$ and

$$[U_0 u_1, u_2]_0 = [u_1, V_0 u_2]_0, \quad u_1, u_2 \in P_0 H^2(\overline{O}, \mathcal{H}).$$

Then, by a generalization of Krein's Lemma (see [2, Lemma 1.1]), U_0 and V_0 can be extended by continuity to bounded operators U and V, respectively, in $\mathcal{K}$ such that $UV = VU = 1$ and

$$[Ux_1, x_2]_{\mathcal{K}} = [x_1, Vx_2]_{\mathcal{K}}, \qquad x_1, x_2 \in \mathcal{K}.$$

The operator U is unitary in the Krein space $\mathcal{K}$.

Assume that $z \notin \overline{O}$. This implies $z, \bar{z}^{-1} \in \rho(U')$ and $z^{-1}, \bar{z} \in \rho(V')$. As the resolvents of U' and V' map ker W into itself, we find $z, \bar{z}^{-1} \in \rho(U_0)$, $z^{-1}, \bar{z} \in \rho(V_0)$, and

$$(U_0 - z)^{-1} = P_0(U' - z)^{-1}|P_0H^2(\overline{O}, \mathcal{H}). \tag{3.25}$$

Moreover, by [2, Corollary 1.2],

$$z,\ \bar{z}^{-1} \in \rho(U).$$

In order to show that a relation of the form (3.22) holds we define an operator $\Gamma_0 \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ by

$$\Gamma_0 :\ \mathcal{H} \ni y \longmapsto (2\sqrt{\pi})^{-1} P_0 \mathbf{1}y.$$

Here $\mathbf{1}y$ denotes the function identically equal to y on a neighbourhood of $\overline{O}$. Let $z \notin \overline{O}$ and $h_z(\zeta) := (4\pi)^{-1}(\zeta + z)(\zeta - z)^{-1}$. Then making use of (2.6) and (3.25) we find, for $x, y \in \mathcal{H}$,

$$\begin{aligned}
[(F(z) - i\mathrm{Im}^+\ F(0))x, y] &= -\int_{\partial O_0} [F(\zeta)h_z(\zeta)x, y](i\zeta)^{-1} d\zeta \\
&= [h_z\mathbf{1}x, \mathbf{1}y]_0 = (4\pi)^{-1}[(U' + z)(U' - z)^{-1}\mathbf{1}x, \mathbf{1}y]_0 \\
&= [P_0(U' + z)(U' - z)^{-1}(2\sqrt{\pi})^{-1}P_0\mathbf{1}x, (2\sqrt{\pi})^{-1}P_0\mathbf{1}y]_0 \\
&= [(U_0 + z)(U_0 - z)^{-1}(2\sqrt{\pi})^{-1}P_0\mathbf{1}x, (2\sqrt{\pi})^{-1}P_0\mathbf{1}y]_{\mathcal{K}} \\
&= [\Gamma_0^+(U + z)(U - z)^{-1}\Gamma_0 x, y].
\end{aligned}$$

This proves (3.22).

In order to prove (3.23) it is sufficient to verify that the set of all functions of the form $z \mapsto z^m x$, $m = 0, \pm 1, \ldots$, $x \in \mathcal{H}$, is total in $H(\overline{O}, \mathcal{H})$. This is a consequence of Cauchy's integral formula and Runge's Theorem. □

Now with the help of Proposition 2.11, Lemma 3.3 and Theorem 3.7 it is not difficult to prove the following theorem.

Theorem 3.8. *Let G be an $\mathcal{L}(\mathcal{H})$-valued operator function definitizable in Ω, let λ_0 be a point of holomorphy of G in $\Omega \cap \mathbf{C}^+$, and let Ω' be a domain in $\overline{\mathbf{C}}$ with the same properties as Ω such that $\overline{\Omega'} \subset \Omega$ and $\lambda_0 \in \Omega'$.*

Then there exists a Krein space $\mathcal{K}$, a selfadjoint relation A in $\mathcal{K}$ definitizable in Ω' and $\Gamma \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ such that the set of all points of holomorphy of G in Ω' coincides with $\widetilde{\rho}(A) \cap \Omega'$,

$$\begin{aligned}
G(\lambda) =& \mathrm{Re}^+\ G(\lambda_0) + \Gamma^+\{\lambda - \mathrm{Re}\ \lambda_0 + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A - \lambda)^{-1}\}\Gamma, \\
&\lambda \in \rho(A) \cap \Omega'
\end{aligned} \tag{3.26}$$

and this representation is Ω'-minimal.

If $F(\psi(\lambda)) = -i\, G(\lambda)$ for $\lambda \in \Omega$, $U := \psi(A)$, $\Gamma_0 := (\mathrm{Im}\, \lambda_0)^{\frac{1}{2}}\Gamma$, then the set of all points of holomorphy of F in $\psi(\Omega')$ coincides with $\rho(U) \cap \psi(\Omega')$,

$$F(z) = i\, \mathrm{Im}^+\, F(0) + \Gamma_0^+(U+z)(U-z)^{-1}\Gamma_0, \quad z \in \rho(U) \cap \psi(\Omega'), \tag{3.27}$$

and this representation is $\psi(\Omega')$-minimal.

Proof. Let Ω'' be a domain in $\overline{\mathbf{C}}$ with the same properties as Ω and Ω' and $\overline{\Omega'} \subset \Omega''$, $\overline{\Omega''} \subset \Omega$ and let Δ be the union of a finite number of pairwise disjoint connected open subsets of $\Omega \cap \overline{\mathbf{R}}$ such that $\overline{\Omega'' \cap \mathbf{R}} \subset \Delta$ and $\overline{\Delta} \subset \Omega \cap \overline{\mathbf{R}}$. In addition, assume that the endpoints of the connected components of Δ are finite and do not belong to $\mathrm{K}(G, \Omega)$ (see Definition 2.5).

Let $G = G_1 + G_2 + G_3$ be a decomposition of G as in Proposition 2.11 with the set denoted by Ω' in Proposition 2.11 replaced by the set Ω'' defined in this proof. Then $G_1 + G_2$ is a definitizable function which is locally holomorphic in $(\overline{\mathbf{R}} \setminus \overline{\Delta}) \cup ((\overline{\mathbf{C}} \setminus \overline{\mathbf{R}}) \setminus \Omega'')$. Let $A_{1,2}$ be a minimal representing selfadjoint relation in a Krein space $\mathcal{K}_{1,2}$ for $G_1 + G_2$ (see [5]). Then the set of all points of holomorphy of $G_1 + G_2$ coincides with $\widetilde{\rho}(A_{1,2})$ and

$$\begin{aligned} G_1(\lambda) + G_2(\lambda) = \mathrm{Re}^+\, &(G_1(\lambda_0) + G_2(\lambda_0)) + \\ &+ \Gamma_{1,2}^+\{\lambda - \mathrm{Re}\, \lambda_0 + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A_{1,2} - \lambda)^{-1}\}\Gamma_{1,2}, \quad \lambda \in \rho(A_{1,2}). \end{aligned} \tag{3.28}$$

The function G_3 is locally holomorphic on Ω''. Let A_3 be a minimal representing selfadjoint relation in a Krein space $\mathcal{K}_3$ for G_3 with $\widetilde{\sigma}(A_3) \subset \overline{\mathbf{C}} \setminus \overline{\Omega'}$, which exists by Theorem 3.7:

$$\begin{aligned} G_3(\lambda) = &\mathrm{Re}^+\, G_3(\lambda_0) + \Gamma_3^+\{\lambda - \mathrm{Re}\, \lambda_0 + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A_3 - \lambda)^{-1}\}\Gamma_3, \\ &\lambda \in \Omega' \setminus \{\infty\}. \end{aligned} \tag{3.29}$$

Let $\mathcal{K} := \mathcal{K}_{1,2} \times \mathcal{K}_3$, let A be the diagonal relation defined by

$$A = \left\{ \begin{pmatrix} (k_{1,2} & k_3)^{\mathrm{T}} \\ (k'_{1,2} & k'_3)^{\mathrm{T}} \end{pmatrix} : \begin{pmatrix} k_{1,2} \\ k'_{1,2} \end{pmatrix} \in A_{1,2},\ \begin{pmatrix} k_3 \\ k'_3 \end{pmatrix} \in A_3 \right\} \tag{3.30}$$

and let $\Gamma := \binom{\Gamma_{1,2}}{\Gamma_3} \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ w.r.t. $\mathcal{K} := \mathcal{K}_{1,2} \times \mathcal{K}_3$. Then $\widetilde{\rho}(A) \cap \Omega' = \widetilde{\rho}(A_{1,2}) \cap \Omega'$ and this set coincides with the set of all points of holomorphy in Ω' of $G_1 + G_2$ and, hence, of G. We have

$$\begin{aligned} G(\lambda) = \mathrm{Re}^+\, G(\lambda_0) + \Gamma^+\{\lambda - \mathrm{Re}\, \lambda_0 + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A - \lambda)^{-1}\}\Gamma, \\ \lambda \in \rho(A) \cap \Omega'. \end{aligned} \tag{3.31}$$

It remains to prove that (3.31) is Ω'-minimal. Let Ω_0 be a domain of $\overline{\mathbf{C}}$ with the same properties as Ω' such that $\overline{\Omega_0} \subset \Omega'$ and $\lambda_0 \in \Omega_0$, and let Δ_0 be a finite union of connected open subsets of $\Omega' \cap \overline{\mathbf{R}}$ such that the endpoints of the components of Δ_0 belong to Ω' and possess open neighbourhoods of positive or of negative type with respect to A (see Theorem 3.6, (i)). If $\widetilde{E}_{1,2} := E(\Delta_0, A_{1,2}) + E(\Omega_0 \setminus \overline{\mathbf{R}}, A_{1,2})$,

$\widetilde{A}_{1,2} := A_{1,2}|\widetilde{E}_{1,2}\mathcal{K}_{1,2}$, then by the minimality of the representation (3.28) the function

$$\lambda \longmapsto (\widetilde{E}_{1,2}\Gamma_{1,2})^{+}\{\lambda - \operatorname{Re}\lambda_0 + (\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(\widetilde{A}_{1,2}-\lambda)^{-1}\}(\widetilde{E}_{1,2}\Gamma_{1,2}) \quad (3.32)$$

is also minimally represented, that is

$$\widetilde{E}_{1,2}\mathcal{K}_{1,2} \\ = \operatorname{closp}\{(1+(\lambda-\lambda_0)(\widetilde{A}_{1,2}-\lambda)^{-1})\widetilde{E}_{1,2}\Gamma_{1,2}y :\ \lambda \in \rho(A_{1,2}) \cap \Omega_0,\ y \in \mathcal{H}\}. \quad (3.33)$$

But in view of $\sigma(A_3) \subset \overline{\mathbf{C}} \setminus \overline{\Omega'}$ we have, for $\widetilde{E} := E(\Delta_0, A) + E(\Omega_0 \setminus \overline{\mathbf{R}}, A)$,

$$\widetilde{E} = \begin{pmatrix} \widetilde{E}_{1,2} & 0 \\ 0 & 0 \end{pmatrix}, \quad \widetilde{E}\Gamma = \begin{pmatrix} \widetilde{E}_{1,2}\Gamma_{1,2} \\ 0 \end{pmatrix}, \quad \text{w.r.t.} \quad \mathcal{K} := \mathcal{K}_{1,2} \times \mathcal{K}_3.$$

Therefore, (3.33) is equivalent to

$$\widetilde{E}\mathcal{K} = \operatorname{closp}\{(1+(\lambda-\lambda_0)(A-\lambda)^{-1})\widetilde{E}\Gamma y :\ \lambda \in \rho(A) \cap \Omega_0,\ y \in \mathcal{H}\},$$

and the representation (3.26) is Ω'-minimal. Theorem 3.8 is proved. □

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